# ADVANCED TOPICS IN FINITE ELEMENT METHOD

## 2D truss structures

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# INTRODUCTION

- 2D trusses are one of the most common types of structures. The structure of a truss is economic since the ratio of the structure weight to forces carried by this structure is expressed as a small number. According to assumptions, loads (concentrated forces) will act on nodes only (temperature loads are an exception here) and connection bars will be joined with nodes in an articulated way.
- Although most structures which have been built lately are trusses with rigid nodes, methods of solving problems in truss statics with articulated joints are still very important in engineering practice. The system of a plane truss with an articulated joint is the simplest example of an construction showing the idea of the finite element method without employing any complicated details.

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# BASIC RELATIONS AND NOTATIONS

- We assume that the bar of a plane truss (we will also call it an element) is straight and homogeneous (it means that it is made from a homogeneous material without fractures and holes and has a constant cross section) and it joins nodes $i$ (the first node) and $j$ (the last node). Notations for these nodes ($i$, $j$) are local notations which are the same for all bars and they are to define element orientation.
- Structure nodes also have global numbers which allow us to identify them. Global numbers are marked as $n_i$ (the global number of the first node) and $n_j$ (the global number of the last node). The node of a plane truss can move on the plane $XY$ only. In mechanics, it means that the node has two degrees of freedom because in order to determine its location during its motion it should be given two coordinates.

# BASIC RELATIONS AND NOTATIONS

- The situation of the node $i$ of a rigid structure will be determined by initial coordinates $X_i$, $Y_i$ with respect to the coordinate system which will be used for the description of the whole structure. This system is global and its axes will be denoted by $X$, $Y$. The location of the node $i$, after its deformation caused by loads, is determined by two components of the displacement vector of nodes $u_{iX}$ and $u_{iY}$.
- This method is called the Langrange description in mechanics. We introduce a local coordinate system $x$, $y$. The $x$ axis of the system overlaps the axis of the bar and has its beginning at the first node of an element $i$, while the $y$ axis is perpendicular to the $x$ axis and is directed in such a way that the $Z$ axis of the global coordinate system and $z$ axis of the local system have the same sense and direction.

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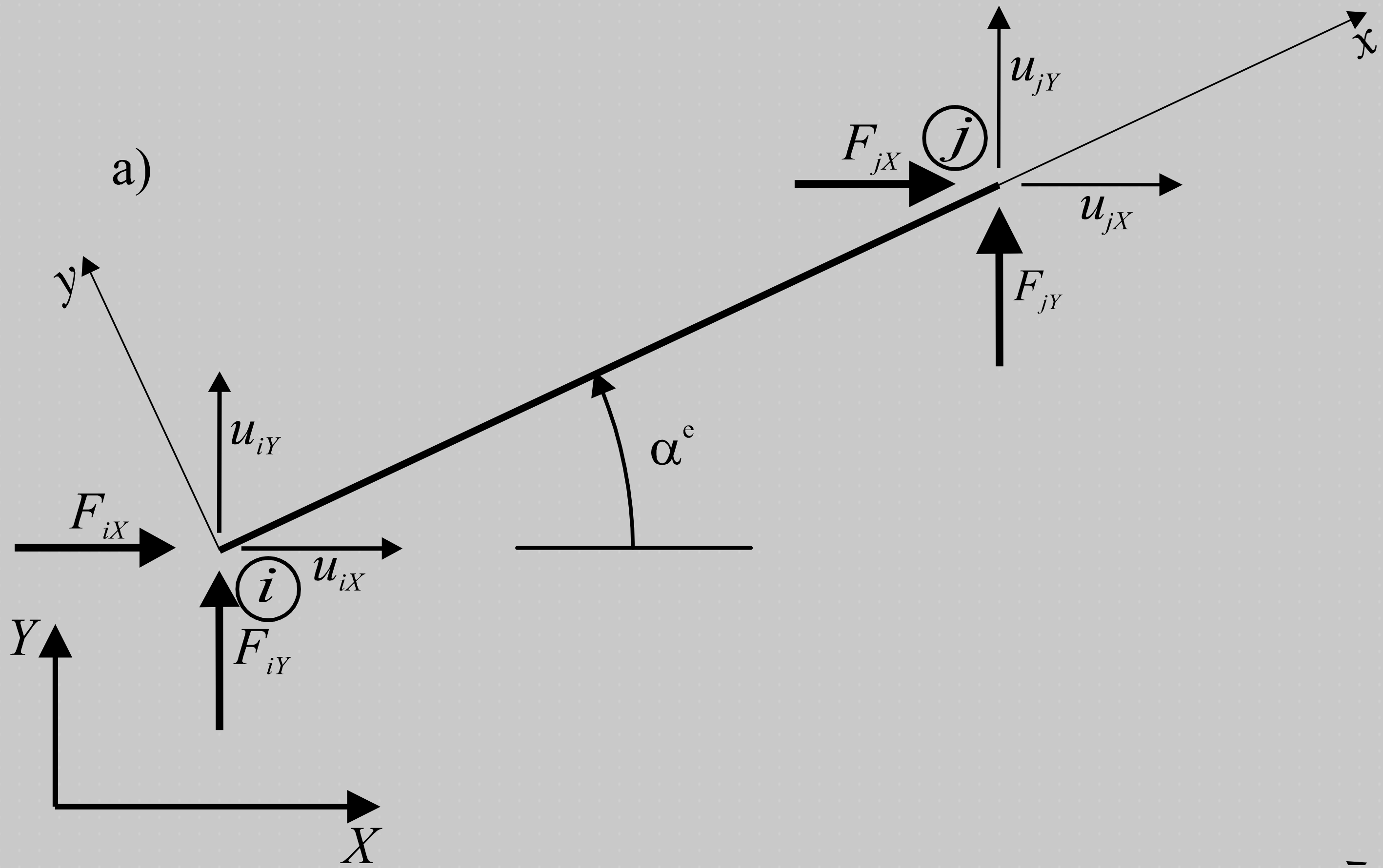
# BASIC RELATIONS AND NOTATIONS




The global coordinate system $XY$

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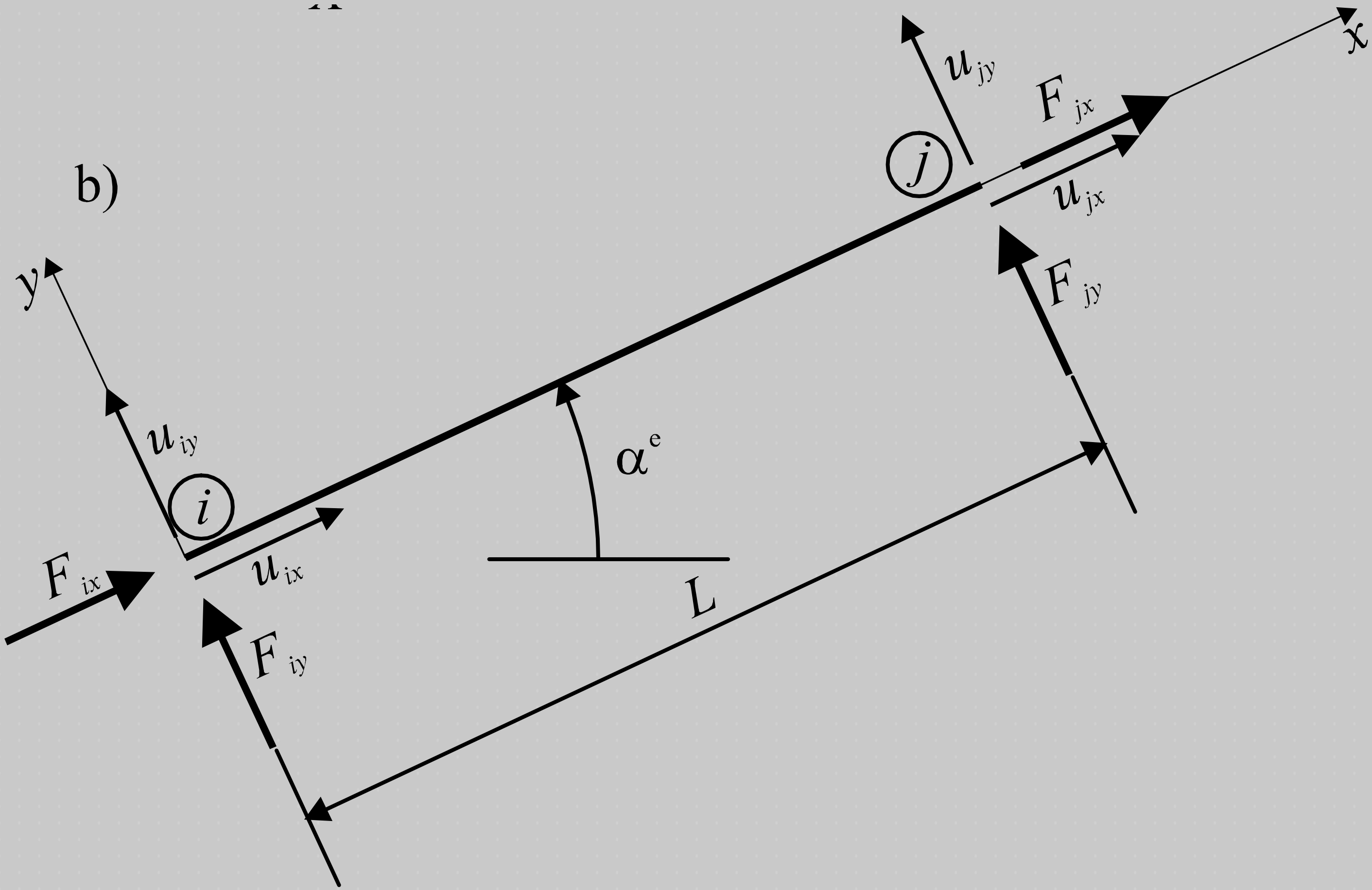
# BASIC RELATIONS AND NOTATIONS




The local coordinate system *xy*

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# BASIC RELATIONS AND NOTATIONS

- Because we accept that both coordinate systems are right-torsion, we can obtain the axis *y* by rotating the *x* axis clockwise through the angle $\pi/2$.
- The most important notations, directions as well as senses of vectors and the coordinate systems are shown in previous figure.
- Nodal displacements and forces of elements are written as column matrices (vectors)

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# BASIC RELATIONS AND NOTATIONS

- The nodal displacement vector of the initial node $i$
- and the end node $j$ in the local coordinate system:

$$\mathbf{u}'_i = \begin{bmatrix} u_{ix} \\ u_{iy} \end{bmatrix} \qquad \mathbf{u}'_j = \begin{bmatrix} u_{jx} \\ u_{jy} \end{bmatrix}$$

- The nodal displacement vector of the element $e$ in the local coordinate system:

$$\mathbf{u}'^e = \begin{bmatrix} \mathbf{u}'_i \\ \mathbf{u}'_j \end{bmatrix} = \begin{bmatrix} u_{ix} \\ u_{iy} \\ u_{jx} \\ u_{jy} \end{bmatrix}$$

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# BASIC RELATIONS AND NOTATIONS

- The nodal forces vector of the initial node $i$ and the end node $j$ in the local coordinate system:

$$\mathbf{f}'_i = \begin{bmatrix} F_{ix} \\ F_{iy} \end{bmatrix} \qquad \mathbf{f}'_j = \begin{bmatrix} F_{jx} \\ F_{jy} \end{bmatrix}$$

- The nodal forces vector of the element $e$ in the local coordinate system:

$$\mathbf{f}'^{e} = \begin{bmatrix} \mathbf{f}'_i \\ \mathbf{f}'_j \end{bmatrix} = \begin{bmatrix} F_{ix} \\ F_{iy} \\ F_{jx} \\ F_{jy} \end{bmatrix}$$

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# THE STIFFNESS MATRIX IN THE LOCAL COORDINATE SYSTEM

- We look for the relation between nodal force vectors and nodal displacement vectors, which is necessary to express equilibrium equations depending on the nodal displacements:

$$\mathbf{K}'^{e}\,\mathbf{u}'^{e} = \mathbf{f}'^{e}$$

- The general method of building such a relationship consists of using the principle of virtual work, but in this case we will apply different approach. We will use the equilibrium conditions in their basic forms which is possible in the case of bar elements.

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# THE STIFFNESS MATRIX IN THE LOCAL COORDINATE SYSTEM

- Equilibrium equations for the element *e* lead to the following relations:

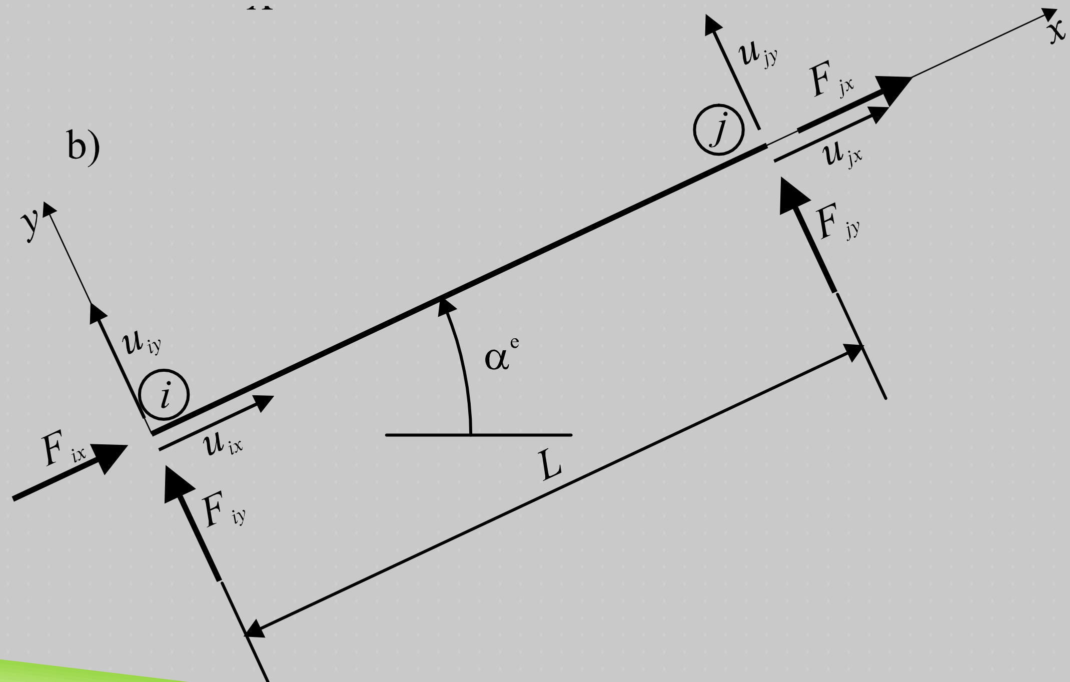
$$\sum F_x = F_{ix} + F_{jx} = 0 \quad \sum F_y = F_{iy} + F_{jy} = 0 \quad \sum M_i = F_{jy}\, L = 0$$

and we obtain

$$F_{iy} = 0$$

$$F_{jy} = 0$$

$$F_{ix} = -F_{jx}$$

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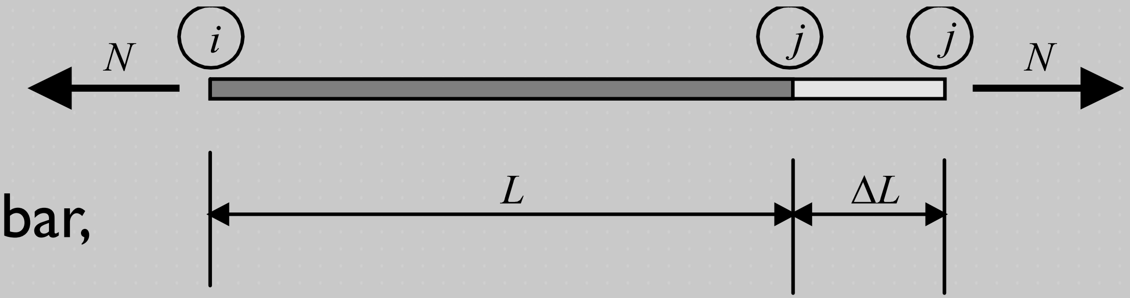
# THE STIFFNESS MATRIX IN THE LOCAL COORDINATE SYSTEM

- Since the set of three equilibrium equations $F_{iy} = 0$, $F_{jy} = 0$, $F_{ix} = -F_{jx}$, contains four unknown parameters, this problem is statically indeterminate. The arrangement of an additional equation is necessary in order to make the determination of nodal forces possible. This equation ought to use the relation between nodal displacements of an element and its internal forces.

# THE STIFFNESS MATRIX IN THE LOCAL COORDINATE SYSTEM

- Hooke's law written for a simple case of axial tension of a straight and homogeneous bar contains these relations:

$$\Delta L = \frac{N\,L}{E\,A}$$




$N$ - the axial force in the bar,

$L$ - the bar length,

$\Delta L$ - increment of the bar length;

$E$ - Young's modulus of the material from which the bar is made

$A$ - the area of the bar cross section.

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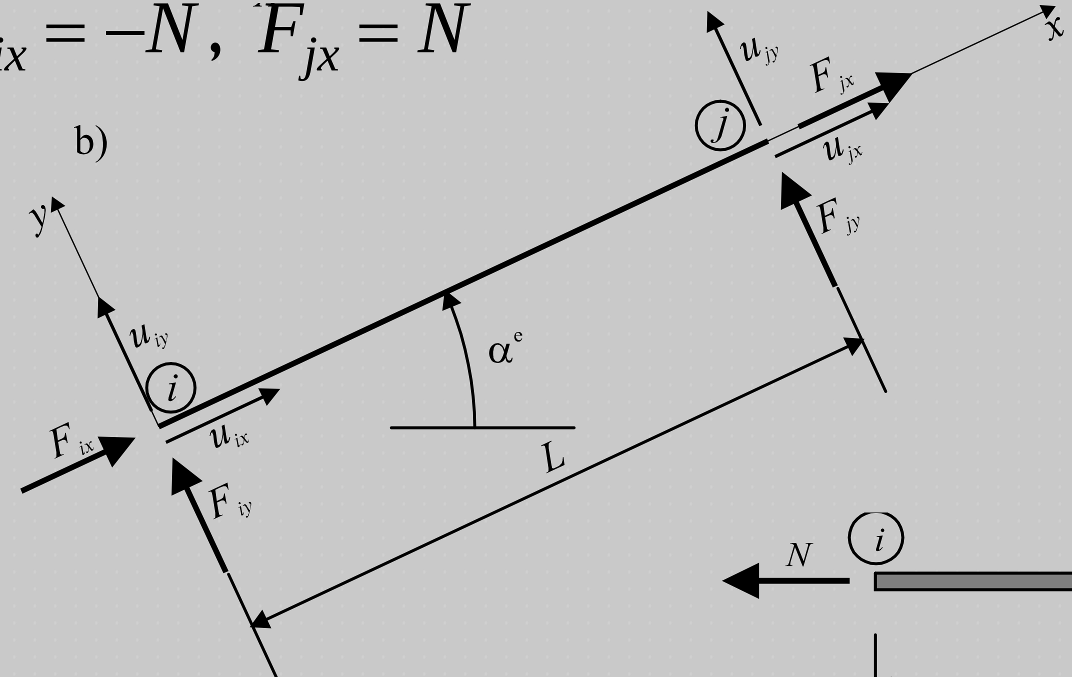
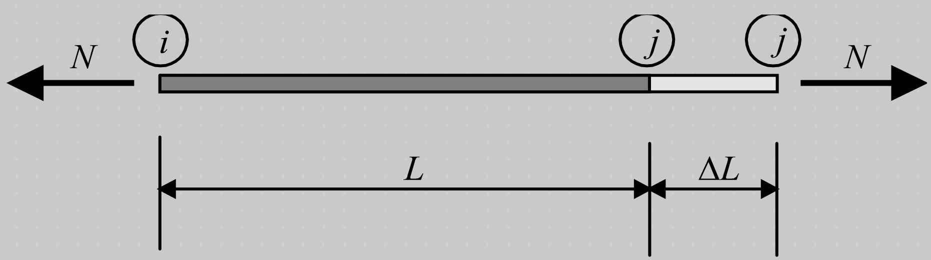
# THE STIFFNESS MATRIX IN THE LOCAL COORDINATE SYSTEM

- We can observe simple relations between nodal forces acting on the bar, that is, $F_{ix}$ , $F_{jx}$ and the axial force $N$:

$$F_{ix} = -N \,, \; F_{jx} = N$$

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# THE STIFFNESS MATRIX IN THE LOCAL COORDINATE SYSTEM

- $F_{ix} = -N$ $F_{jx} = N$ these relations satisfy the equilibrium equation identically: $F_{ix} = -F_{jx}$

- The increment of the bar length due to tension results from axial displacements of the bar endings:

$$\Delta L = u_{jx} - u_{ix}$$

which leads to the relation: $$N = \frac{EA}{L}\left(u_{jx} - u_{ix}\right)$$

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# THE STIFFNESS MATRIX IN THE LOCAL COORDINATE SYSTEM

- Taking into consideration the relationship between the axial force of the element and nodal forces

$$F_{ix} = -N, \; F_{jx} = N$$

with respect to $N = \dfrac{EA}{L}\left(u_{jx} - u_{ix}\right)$

we obtain:

$$F_{ix} = \frac{EA}{L}\left(u_{ix} - u_{jx}\right) \qquad F_{iy} = 0$$

$$F_{jx} = \frac{EA}{L}\left(-u_{ix} + u_{jx}\right) \qquad F_{jy} = 0$$

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# THE STIFFNESS MATRIX IN THE LOCAL COORDINATE SYSTEM

- The resulting relations are the searched relations $\mathbf{K}'^{e}\,\mathbf{u}'^{e} = \mathbf{f}'^{e}$ between the nodal forces and nodal displacements of the truss element:

$$
\left[\begin{array}{c:c:c:c}
\frac{EA}{L} & 0 & -\frac{EA}{L} & 0 \\
\hdashline
0 & 0 & 0 & 0 \\
\hdashline
-\frac{EA}{L} & 0 & \frac{EA}{L} & 0 \\
\hdashline
0 & 0 & 0 & 0
\end{array}\right]
\begin{bmatrix} u_{ix} \\ u_{iy} \\ u_{jx} \\ u_{jy} \end{bmatrix}
=
\begin{bmatrix} F_{ix} \\ F_{iy} \\ F_{jx} \\ F_{jy} \end{bmatrix}.
$$

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# THE STIFFNESS MATRIX IN THE LOCAL COORDINATE SYSTEM

- After considering notations $\mathbf{u}'^{e}$ , $\mathbf{f}'^{e}$ and $\mathbf{K}'^{e}\,\mathbf{u}'^{e} = \mathbf{f}'^{e}$ the above form leads to the equation:

$$\mathbf{K}'^{e} = \begin{bmatrix} \frac{EA}{L} & 0 & -\frac{EA}{L} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{EA}{L} & 0 & \frac{EA}{L} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

which defines a matrix $\mathbf{K}'^{e}$.

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# THE STIFFNESS MATRIX IN THE LOCAL COORDINATE SYSTEM

- This matrix will be called the element stiffness matrix of a plane truss. The matrix in the previous form expresses relationships between the vector  and the nodal force vector of an element  in the local coordinate system.
- The stiffness matrix  can be simplified to:

$$\mathbf{K}'^e = \begin{bmatrix} \mathbf{J}' & -\mathbf{J}' \\ -\mathbf{J}' & \mathbf{J}' \end{bmatrix}$$

where **J'** is the square matrix defined in the following way:

$$\mathbf{J}' = \frac{EA}{L}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

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# COORDINATE SYSTEM ROTATION

- The form of the element stiffness matrix determined in the local coordinate system will not be convenient in further considerations for which we will use matrices of different elements. The most convenient method is transforming all matrices to the form which is defined in one common coordinate system. Such a system will be called the ***global coordinate system***.
- It can be the system of a certain type: cartesian, polar or curvilinear. The cartesian coordinate system is the most convenient system for a truss.
- Nodal coordinates of a structure are usually given in the global coordinate system.

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# COORDINATE SYSTEM ROTATION

- Now we convert the element stiffness matrix to the global system. We start the transformations by finding relationships for a single node:

$$u_{iX} = u_{ix}\cos\alpha - u_{iy}\sin\alpha \qquad u_{iY} = u_{ix}\sin\alpha + u_{iy}\cos\alpha$$

or in matrix form: $\begin{bmatrix} u_{iX} \\ u_{iY} \end{bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} u_{ix} \\ u_{iy} \end{bmatrix}$

where $c = \cos\alpha$ and $s = \sin\alpha$ .

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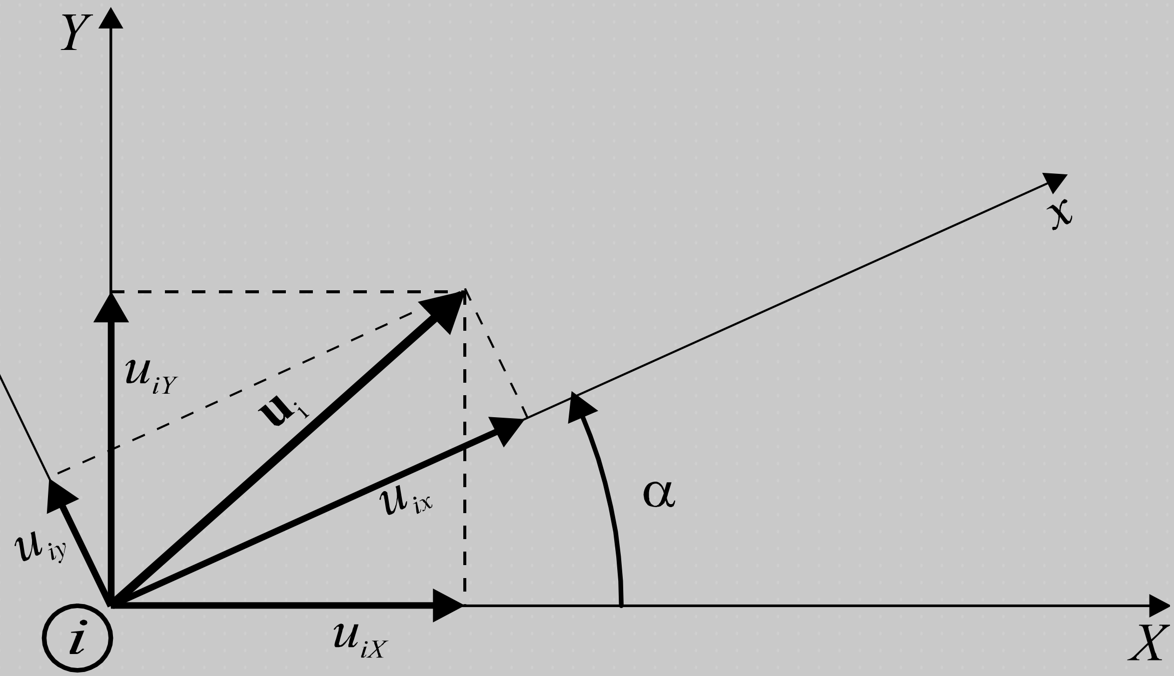
# COORDINATE SYSTEM ROTATION




Displacement vector components in the global and local coordinate systems rotated through the $\alpha$ angle

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# COORDINATE SYSTEM ROTATION

- Denoting and taking into consideration:

$$\mathbf{u}_i = \begin{bmatrix} u_{iX} \\ u_{iY} \end{bmatrix} \qquad \mathbf{u}'_i = \begin{bmatrix} u_{ix} \\ u_{iy} \end{bmatrix} \qquad \mathbf{u}'_j = \begin{bmatrix} u_{jx} \\ u_{jy} \end{bmatrix}$$

we obtain: $\mathbf{u}_i = \mathbf{R}_i \mathbf{u}'_i$

$$\mathbf{R}_i = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$$

- the transformation matrix of the vector from the local to global coordinate system.

# COORDINATE SYSTEM ROTATION

- A reverse relation will be required:

$$\mathbf{u}'_i = \left(\mathbf{R}_i\right)^{-1} \mathbf{u}_i$$

where $(\mathbf{R}_i)^{-1}$ is the inverse matrix of $\mathbf{R}_i$; it means that it has such a property that $\mathbf{R}_i \left(\mathbf{R}_i\right)^{-1} = \mathbf{I}$ , where $\mathbf{I}$ is the identity matrix: $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

- The matrix $\mathbf{R}_i$ like other „rotation matrices" has a property that gives: $\left(\mathbf{R}_i\right)^{-1} = \left(\mathbf{R}_i\right)^{\mathsf{T}}$

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# COORDINATE SYSTEM ROTATION

- It means that $\mathbf{R}_i$ is the orthogonality matrix (the determinant of this matrix is equal to 1, i.e. $\det(\mathbf{R}_i)=1$; $\det(\mathbf{R}_i)^{\mathsf{T}}=1$).
- We can easily check the upper property of the matrix $\mathbf{R}_i$ by making a direct calculation:

$$\mathbf{R}_i\left(\mathbf{R}_i\right)^{\mathsf{T}}=\begin{bmatrix} c & -s \\ s & c \end{bmatrix}\cdot\begin{bmatrix} c & s \\ -s & c \end{bmatrix}=\begin{bmatrix} c^2+s^2 & cs-sc \\ sc-cs & c^2+s^2 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=\mathbf{I}$$

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# COORDINATE SYSTEM ROTATION

- The transformation matrix contains the blocks of the nodal transformation matrix:

$$\mathbf{R}^e = \begin{bmatrix} \mathbf{R}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_j \end{bmatrix}$$

- $\mathbf{R}_i$ and $\mathbf{R}_j$ are the transformation matrices of the first and last node, $\mathbf{0}$ is the part of the matrix containing zero values.
- $\mathbf{R}_i$ and $\mathbf{R}_j$ are usually identical (for straight elements) because rotation angles of the vector are equal.

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# COORDINATE SYSTEM ROTATION

- Finally, the relationships between the nodal displacement vector of the element expressed in the local system and the same vector in the global system have the form:

$$\mathbf{u}^e = \mathbf{R}^e \mathbf{u}'^e$$

$$\mathbf{u}'^e = \left(\mathbf{R}^e\right)^\mathsf{T} \mathbf{u}^e$$

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# COORDINATE SYSTEM ROTATION

- The relationship between the nodal force vector of an element in the local system and the same vector in the global system is identical to the relationship that we have obtained in the equations describing displacements:

$$\mathbf{f}_i = \mathbf{R}_i \mathbf{f}'_i \qquad \mathbf{f}^e = \mathbf{R}^e \mathbf{f}'^e$$

$$\mathbf{f}'_i = \left(\mathbf{R}_i\right)^{\mathsf{T}} \mathbf{f}_i \qquad \mathbf{f}'^e = \left(\mathbf{R}^e\right)^{\mathsf{T}} \mathbf{f}^e$$

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# STIFFNESS MATRIX IN THE GLOBAL COORDINATE SYSTEM

- Multiplying $\mathbf{K}'^{e}\,\mathbf{u}'^{e} = \mathbf{f}'^{e}$ by the transformation matrix and substituting relation $\mathbf{u}'^{e} = \left(\mathbf{R}^{e}\right)^{\mathsf{T}}\mathbf{u}^{e}$, we obtain:

$$\mathbf{R}^{e}\mathbf{K}'^{e}\left(\mathbf{R}^{e}\right)^{\mathsf{T}}\mathbf{u}^{e} = \mathbf{R}^{e}\mathbf{f}'^{e}$$

- On the basis of relation $\mathbf{f}^{e} = \mathbf{R}^{e}\mathbf{f}'^{e}$ the right hand side of this equation is equal to $\mathbf{f}^{e}$, so if we introduce the notation $\mathbf{K}^{e} = \mathbf{R}^{e}\mathbf{K}'^{e}\left(\mathbf{R}^{e}\right)^{\mathsf{T}}$ we obtain: $\mathbf{f}^{e} = \mathbf{K}^{e}\mathbf{u}^{e}$
- It is the required relationship between nodal forces and displacements of the element in the global coordinate system.

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# STIFFNESS MATRIX IN THE GLOBAL COORDINATE SYSTEM

- If we perform the multiplication in $\mathbf{K}^e = \mathbf{R}^e \mathbf{K}'^e \left(\mathbf{R}^e\right)^{\mathsf{T}}$, we obtain:

$$\mathbf{K}^e = \begin{bmatrix} \mathbf{J} & -\mathbf{J} \\ -\mathbf{J} & \mathbf{J} \end{bmatrix} \quad \mathbf{J} = \frac{EA}{L}\begin{bmatrix} c^2 & sc \\ sc & s^2 \end{bmatrix} \quad \begin{aligned} c &= \cos\alpha = L_X / L \\ s &= \sin\alpha = L_Y / L \end{aligned}$$

- 

- Now we can exchange form of the $\mathbf{J}$ matrix into equivalent one in which trigonometric functions do not exist:

$$\mathbf{J} = \frac{EA}{L^3}\begin{bmatrix} L_X^2 & L_X L_Y \\ L_X L_Y & L_Y^2 \end{bmatrix}$$

# EQUILIBRIUM EQUATIONS AND AGGREGATION

- Replacing existing bars (elements) of a truss by nodal forces we obtain a group of nodes which can be treated as material particles with two degrees of freedom. These nodes are loaded with concentrated forces coming from elements or external loads. The equilibrium conditions for such a node are as follows:

$$\sum P_X = \sum_{k=1}^{E_n} \left( - F_{nX}^{e_k} \right) + P_{nX} = 0 \qquad \sum P_Y = \sum_{k=1}^{E_n} \left( - F_{nY}^{e_k} \right) + P_{nY} = 0$$

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# EQUILIBRIUM EQUATIONS AND AGGREGATION

$$\sum P_X = \sum_{k=1}^{E_n}\left(-F_{nX}^{e_k}\right) + P_{nX} = 0 \qquad \sum P_Y = \sum_{k=1}^{E_n}\left(-F_{nY}^{e_k}\right) + P_{nY} = 0$$

$F_{nX}^{e_k}$ - component in the direction $X$ of nodal forces from the element numbered $e_k$ acting on a node $n$,

$P_{nX}$ - component in the direction $X$ of the external forces acting on the node $n$,

$E_n$ - number of elements joined to the node $n$.

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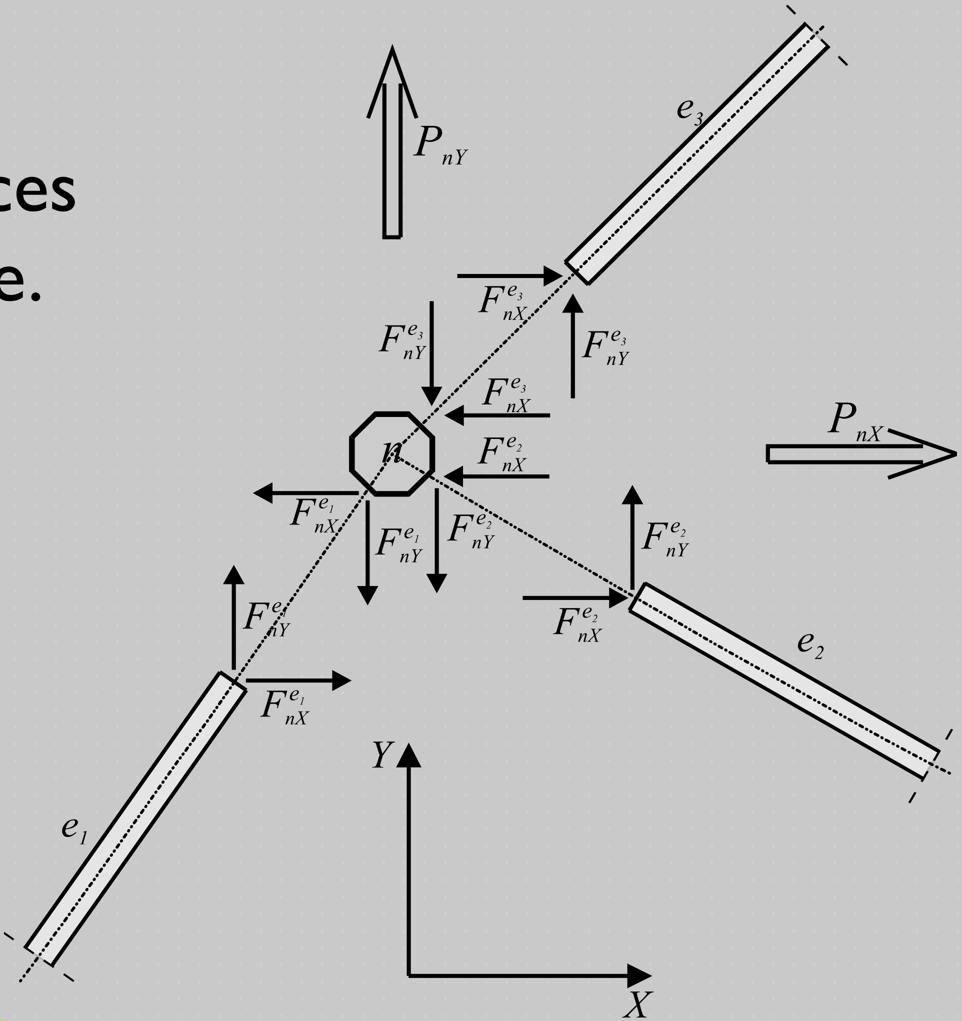
# EQUILIBRIUM EQUATIONS AND AGGREGATION

Nodal and external forces acting on the truss node.

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# EQUILIBRIUM EQUATIONS AND AGGREGATION

$$\sum P_X = \sum_{k=1}^{E_n} \left( -F_{nX}^{e_k} \right) + P_{nX} = 0 \qquad \sum P_Y = \sum_{k=1}^{E_n} \left( -F_{nY}^{e_k} \right) + P_{nY} = 0$$

- Now we transform the set of this equtions to the form containing nodal displacements:

$$\begin{bmatrix} \mathbf{K}_{1n} & \mathbf{K}_{2n} & \dots & \mathbf{K}_{in} & \dots & \mathbf{K}_{N_n n} \end{bmatrix} \mathbf{u} = \mathbf{p}_n$$

$$\mathbf{p}_n = \begin{bmatrix} P_{nX} \\ P_{nY} \end{bmatrix}$$

the vector of external forces acting on the node *n*

global vector of nodal displacements of a structure

$$\mathbf{u} = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_i \\ \vdots \\ \mathbf{u}_{N_n} \end{bmatrix}$$

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# EQUILIBRIUM EQUATIONS AND AGGREGATION

$$\begin{bmatrix} \mathbf{K}_{1n} & \mathbf{K}_{2n} & \dots & \mathbf{K}_{in} & \dots & \mathbf{K}_{N_n n} \end{bmatrix} \mathbf{u} = \mathbf{p}_n$$

- matrices $\mathbf{K}_{in}$ are quadratic matrices with dimensions 2x2 determined as follows: $\mathbf{K}_{nn} = \sum_{k=1}^{E_n} \mathbf{J}^{e_k}$ if $i=n$ ,

  $e_1, e_2 \dots e_k \dots e_{En}$ are numbers of the elements joined at node $n$,
- if $i \neq n$ and nodes $i$ and $n$ are connected by some element with a number $e$, then $\mathbf{K}_{in} = -\mathbf{J}^e$

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# EQUILIBRIUM EQUATIONS AND AGGREGATION

- Arranging $\begin{bmatrix}\mathbf{K}_{1n} & \mathbf{K}_{2n} & \dots & \mathbf{K}_{in} & \dots & \mathbf{K}_{N_n n}\end{bmatrix}\mathbf{u} = \mathbf{p}_n$ for all nodes we obtain the final form allowing determination of nodal displacements: $\mathbf{Ku} = \mathbf{p}$

$$
\begin{bmatrix}
\mathbf{K}_{11} & \mathbf{K}_{12} & \dots & \mathbf{K}_{1n} & \dots & \mathbf{K}_{1N_n} \\
\mathbf{K}_{21} & \mathbf{K}_{22} & \dots & \mathbf{K}_{2n} & \dots & \mathbf{K}_{2N_n} \\
\vdots & \vdots & \ddots & \vdots & & \vdots \\
\mathbf{K}_{n1} & \mathbf{K}_{n2} & \dots & \mathbf{K}_{nn} & \dots & \mathbf{K}_{nN_n} \\
\vdots & \vdots & & \vdots & \ddots & \vdots \\
\mathbf{K}_{N_n 1} & \mathbf{K}_{N_n 2} & \dots & \mathbf{K}_{N_n n} & \dots & \mathbf{K}_{N_n N_n}
\end{bmatrix}
\begin{bmatrix}
\mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_n \\ \vdots \\ \mathbf{u}_{N_n}
\end{bmatrix}
=
\begin{bmatrix}
\mathbf{p}_1 \\ \mathbf{p}_2 \\ \vdots \\ \mathbf{p}_n \\ \vdots \\ \mathbf{p}_{N_n}
\end{bmatrix}
$$

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# EQUILIBRIUM EQUATIONS AND AGGREGATION

- The matrix $\mathbf{K}$ of the set of equation $\mathbf{Ku} = \mathbf{p}$ is the global stiffness matrix of the structure, the vector $\mathbf{u}$ is the global vector of nodal displacements of the structure and the vector $\mathbf{p}$ is the global vector of nodal forces of the structure.
- Careful numbering of the nodes can allow $\mathbf{K}$ to the banded matrix which is characterised by a fact that non-zero components appear on the main diagonal and closely to it.

$$K_{ij} = K_{ji} \qquad \mathbf{K} = \mathbf{K}^{\mathsf{T}}$$

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# EQUILIBRIUM EQUATIONS AND AGGREGATION

- The matrix $\mathbf{K}$ is a symmetric matrix which means that its components satisfy equations:

$$K_{ij} = K_{ji} \qquad \mathbf{K} = \mathbf{K}^{\mathsf{T}}$$

result from the principle of virtual work.

- Components $K_{nn}$ which are on the main diagonal are always positive ( $K_{nn} > 0$ ) which is a direct conclusion drawn from definitions:

$$\mathbf{J} = \frac{EA}{L}\begin{bmatrix} c^2 & sc \\ sc & s^2 \end{bmatrix} \qquad \mathbf{K}_{nn} = \sum_{k=1}^{E_n} \mathbf{J}^{e_k}$$

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# EQUILIBRIUM EQUATIONS AND AGGREGATION

- The zero component $K_{nn}$ demonstrates geometric changability of a structure and should be removed by a suitable change of a geometric scheme. The matrix $\mathbf{K}$ in $\mathbf{Ku=p}$ is a singular matrix ($|\mathbf{K}| = 0$), hence the set of equation $\mathbf{Ku=p}$ cannot be solved without modifying it. This modification will depend on the consideration of boundary conditions. We will consider this problem in the next section.

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# EQUILIBRIUM EQUATIONS AND AGGREGATION

- The process of building the global stiffness matrix is called aggregation of a matrix. It can be done by means of the method described in previous presentation demanding formation of connection matrices. Since these matrices are large, then their use is not convenient and they are rarely used in computer implementation of the FEM algorithm.
- The method of summation of blocks shown by

$$\begin{bmatrix}\mathbf{K}_{1n} & \mathbf{K}_{2n} & \dots & \mathbf{K}_{in} & \dots & \mathbf{K}_{N_n n}\end{bmatrix}\mathbf{u} = \mathbf{p}_n \quad \text{and} \quad \mathbf{K}_{nn} = \sum_{k=1}^{E_n} \mathbf{J}^{e_k}$$

is much simpler.

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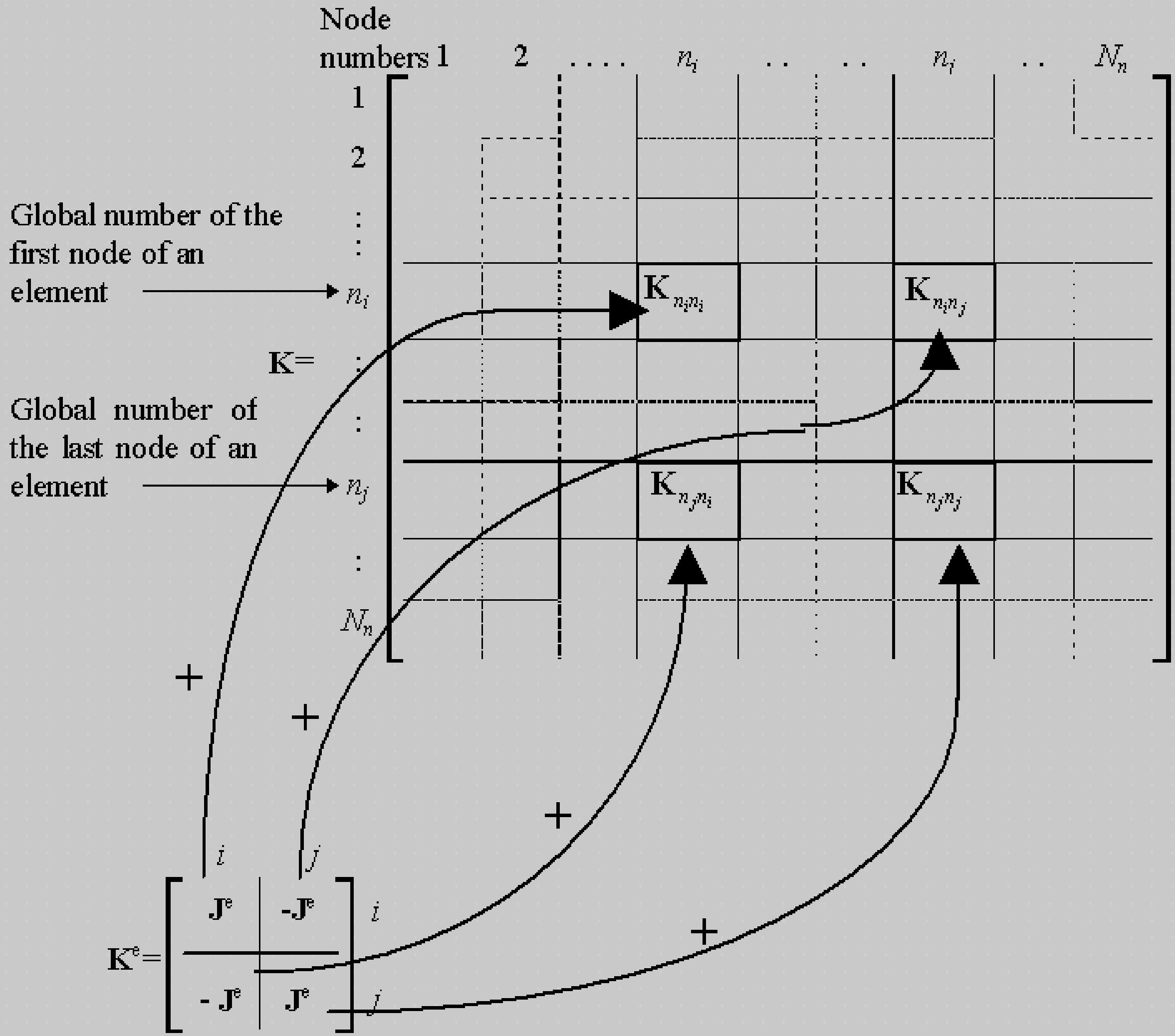
# EQUILIBRIUM EQUATIONS AND AGGREGATION

- The stiffness matrix aggregation scheme

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# EQUILIBRIUM EQUATIONS AND AGGREGATION

- ‘+’ signs located at arrows pointing to the place of location of blocks $\mathbf{K}^e$ mean that blocks $\mathbf{J}^e$ should be added to the existing contents of ‘cells’ of matrices $\mathbf{K}_{n_i n_i}$ or $\mathbf{K}_{n_j n_j}$, and blocks $-\mathbf{J}^e$ lying beyond the diagonal should be added to ‘cells’ $\mathbf{K}_{n_i n_j}$ or $\mathbf{K}_{n_j n_i}$.

In the case of a truss where nodes are usually joined by one element, blocks lying beyond the main diagonal contain only a single matrix $-\mathbf{J}^e$. But blocks lying on the main diagonal contain sums of as many matrices $\mathbf{J}^e$ as elements joined with the node $n_i$.

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# BOUNDARY CONDITIONS

- The global stiffness matrix of a structure is most often a singular matrix directly after the aggregation. It means that the determinant of this matrix is equal to zero. Because the set of $\mathbf{Ku=p}$ has to have only one solution for static problems, we have to modify the global stiffness matrix. It should be done in such a way that the solution of the set of linear this equation is possible.
- The reason for the singularity of the matrix $\mathbf{K}$ is the lack of information about supports of the construction, thus we need to define what the support of the node is.

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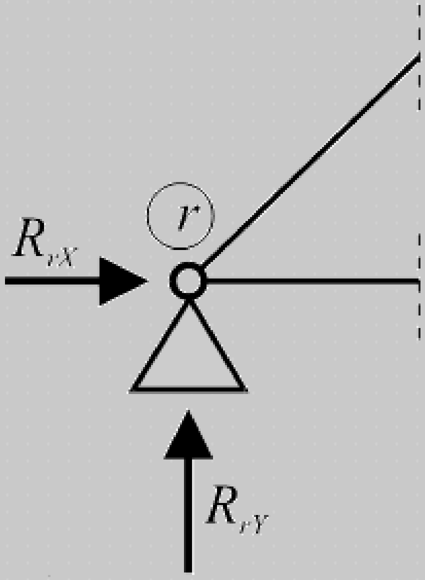
# BOUNDARY CONDITIONS

- For trusses there are two types of supports possible: an articulated support and an articulated movable support.
- The articulated support prevents movements of a node in any direction which means: $u_{rX} = 0 \quad u_{rY} = 0$
- The movement of the support node $r$ causes reactions in two components: $R_X$ and $R_Y$, which counteract the movement of the node $r$.
- This support assures *free support* of a node.

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# BOUNDARY CONDITIONS

- The next support is called an articulated movable support and it prevents movements of a node along one line only, but it allows movement of a node in perpendicular direction with respect to this line. The reaction occurring in the support can have the direction of this line only. It can appear in a few forms, two most often occurring variants give very simple support conditions.

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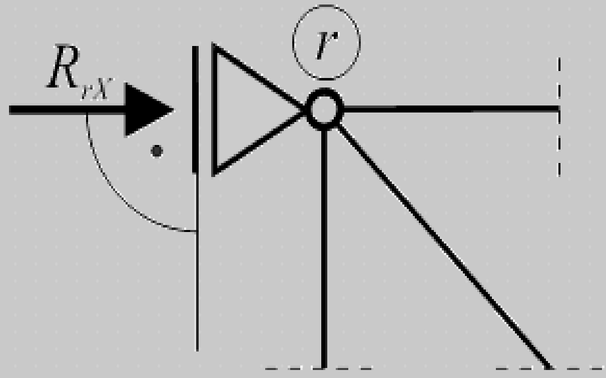
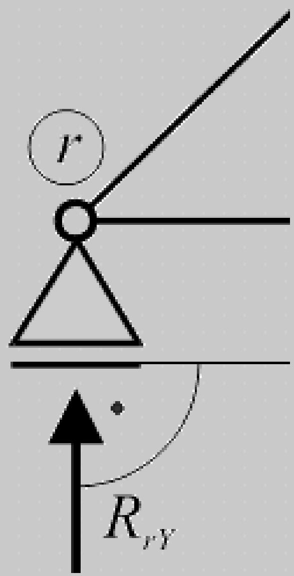
# BOUNDARY CONDITIONS

- support with the possibility of movement along the $Y$ axis of the global coordinate system:

$$\mathbf{u}_{rX} = 0$$




- or along the $X$ axis:

$$\mathbf{u}_{rY} = 0$$

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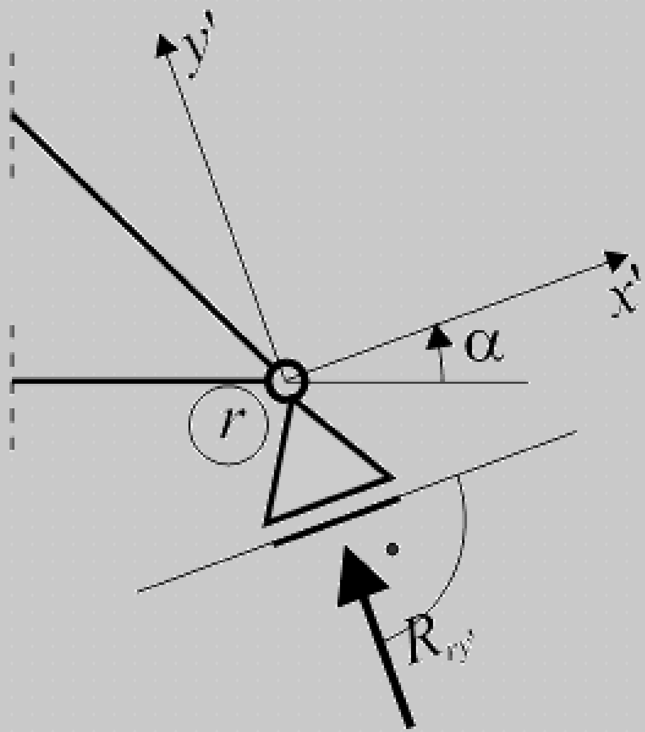
# BOUNDARY CONDITIONS

- The third variant of a movable support causes problems when describing the boundary conditions because the direction of the reaction of this support is not parallel to any axis of the global coordinate system.

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# BOUNDARY CONDITIONS

- It is important because equilibrium equations:

$$\sum P_X = \sum_{k=1}^{E_n} \left( - F_{nX}^{e_k} \right) + P_{nX} = 0 \qquad \sum P_Y = \sum_{k=1}^{E_n} \left( - F_{nY}^{e_k} \right) + P_{nY} = 0$$

leading to **Ku=p** were written in the global coordinate system. In a support with movement not parallel to any axis of the global coordinate system (skew supports) we have to write the boundary conditions in the system $x'y'$ connected with the support. It is rotated with respect to the global system by an angle $\alpha'$.

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# BOUNDARY CONDITIONS

- We will explain the transformation method for a set of equations at a support node to the local system in the next section. Now we will focus on describing the boundary condition. We write the condition of absence of a movement along the $y'$ axis:

$$\mathbf{u}_{ry'} = 0$$

- Equations $u_{rX} = 0$, $u_{rY} = 0$, $\mathbf{u}_{rX} = \mathbf{0}$, $\mathbf{u}_{rY} = \mathbf{0}$ and $\mathbf{u}_{ry'} = \mathbf{0}$ describing the boundary conditions give us the values of displacements at support nodes.

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# BOUNDARY CONDITIONS

- Hence some equations of set **Ku=p** should be removed, because they contain unknown forces acting on support nodes (constraint reactions).
- These equations can be replaced by equations of boundary conditions. It is usually done by modifying some equations from system **Ku=p**.
- Let $m$ be the global number of the degree of freedom which is eliminated by the boundary condition: $u_m = 0$, then we modify the row with the number $m$ in the global stiffness matrix **K**, replacing it by a row containing zeros and the value $1$ in the column $m$ in the next slide.

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# BOUNDARY CONDITIONS

$$\begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} & \cdots & \mathbf{K}_{1m} & \cdots & \mathbf{K}_{1N_n} \\ \mathbf{K}_{21} & \mathbf{K}_{22} & \cdots & \mathbf{K}_{2m} & \cdots & \mathbf{K}_{2N_n} \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ 0 & 0 & & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ \mathbf{K}_{N_n 1} & \mathbf{K}_{N_n 2} & \cdots & \mathbf{K}_{N_n m} & \cdots & \mathbf{K}_{N_n N_n} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \\ \vdots \\ u_{N_n} \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ 0 \\ \vdots \\ P_{N_n} \end{bmatrix}$$

or $\mathbf{K}^r \mathbf{u} = \mathbf{p}^r$

- The nodal load vector $\mathbf{p}$ should be modified so that equation $m$ contains zero on the right side.
- The modified matrices are marked by $r$.

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# BOUNDARY CONDITIONS

- These changes in the stiffness matrix disturb the symmetry because but when (comp. $\mathbf{K}^r\mathbf{u}=\mathbf{p}^r$). The absence of symmetry in the stiffness matrix does not prevent the solving of the equilibrium $\mathbf{Ku}=\mathbf{p}$ but it considerably loads the computer memory storing coefficients $K_{ij}$ either in the core memory (RAM) or external space (disk) which lengthens the solution time for a set of equations.

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# BOUNDARY CONDITIONS

- Thus, let us try to restore the symmetry of the matrix $\mathbf{K}^r$. Let us note that the terms located in the column with the number $m$ are multiplied by the zero value of the displacement $u_m$. Hence we can insert zeros instead of coefficients in the column $m$ (except for one coefficient in the row $m$ which has to be equal to 1).

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# BOUNDARY CONDITIONS

- If we modify the stiffness matrix in that way, the solution of our problem will be the same and the matrix will be a symmetric one:

$$\mathbf{K}^{r} = \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} & \dots & 0 & \dots & \mathbf{K}_{1N_n} \\ \mathbf{K}_{21} & \mathbf{K}_{22} & \dots & 0 & \dots & \mathbf{K}_{2N_n} \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ \mathbf{K}_{N_n 1} & \mathbf{K}_{N_n 2} & \dots & 0 & \dots & \mathbf{K}_{N_n N_n} \end{bmatrix}$$

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# BOUNDARY CONDITIONS

- Finally, we solve the problem: $\mathbf{K}^r\mathbf{u}=\mathbf{p}^r$
- The matrix $\mathbf{K}^r$ is symmetrical and is not singular which means that $\det(\mathbf{K}^r) \neq 0$, if we have properly chosen the boundary conditions.
- On the theorem about the value of a strain energy:

$$(\mathbf{u}^e)^{\mathsf{T}}\mathbf{f}^e + \int_{\mathscr{A}} (\mathbf{N}^e\mathbf{u}^e)^{\mathsf{T}}\mathbf{q}\, d\mathscr{A} = \int_{\mathscr{V}} (\mathbf{B}^e\mathbf{u}^e)^{\mathsf{T}}\left[\mathbf{D}(\mathbf{B}^e\mathbf{u}^e - \varepsilon_o) + \sigma_o\right] d\mathscr{V}$$

  we can conclude that the matrix has to be positive-define, then $\det(\mathbf{K}^r) > 0$.
- Hence the set $\mathbf{K}^r\mathbf{u}=\mathbf{p}^r$ has one solution.

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# BOUNDARY CONDITIONS

- In small finite element systems (programs) the matrix $\mathbf{K}^r$ is usually left in the form noted proviously.
- Large and complex systems used to solve problems described by many thousands of equations usually remove rows and columns containing zeros from $\mathbf{K}^r$ and $\mathbf{p}^r$. This is done to reduce the dimensions of a solved problem. This method of modification of requires re-numbering of degrees of freedom of a structure. Because it is not strictly joined with FEM, we will not describe it here.

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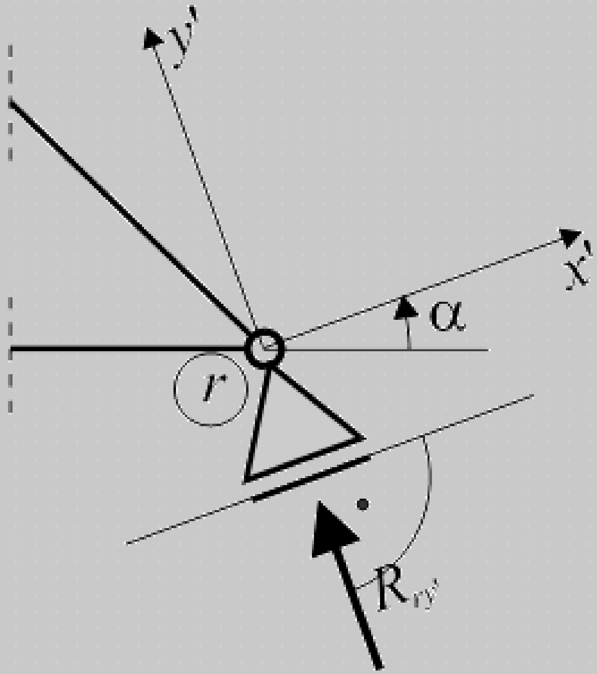
# STIFFNESS MATRIX FOR A 'SKEW' SUPPORT

- Now we are explaining ways of transforming an element stiffness matrix joined to a support node by means of a 'skew' support .

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# STIFFNESS MATRIX FOR A 'SKEW' SUPPORT

- We choose the coordinate system $x'y'$ in such a way that the direction of a support reaction covers the $y'$ axis and the movement will be parallel to the $x'$ axis.
- An alternative choice of the local coordinate system is also possible.
- The $x'$ axis is rotated with respect to the $X$ axis of the global system by the angle $\alpha'$ which we will deem to be positive when the rotation from the $X$ axis to the $x'$ axis is anticlockwise.

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# STIFFNESS MATRIX FOR A 'SKEW' SUPPORT

- If we write equilibrium equations for the support node $r$ in the system $x'y'$, then the boundary condition of this support is determined by equation $\mathbf{u}_{ry'} = 0$ .
- Let us try to perform the necessary transformation. We make use of relations $\mathbf{u}_i = \mathbf{R}_i \mathbf{u}'_i$ and $\mathbf{u}'_i = \left(\mathbf{R}_i\right)^{-1} \mathbf{u}_i$ to pass from the local system of an element to the global one.

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# STIFFNESS MATRIX FOR A 'SKEW' SUPPORT

- Then we express the nodal forces vector at the node *r* as follows:

$$\begin{bmatrix} F_{rx'} \\ F_{ry'} \end{bmatrix} = \begin{bmatrix} c' & s' \\ -s' & c' \end{bmatrix} \begin{bmatrix} F_{rX} \\ F_{rY} \end{bmatrix}$$

or in an abbreviated form:

$$\mathbf{f}'_r = \left(\mathbf{R}'_r\right)^{\mathsf{T}} \mathbf{f}_r$$

# STIFFNESS MATRIX FOR A 'SKEW' SUPPORT

- Next we transform the nodal displacements vector of the support node from the local system to the global one as follows:

$$\begin{bmatrix} u_{rX} \\ u_{rY} \end{bmatrix} = \begin{bmatrix} c' & -s' \\ s' & c' \end{bmatrix} \begin{bmatrix} u_{rx'} \\ u_{ry'} \end{bmatrix}$$

or in a close form:

$$\mathbf{u}_r = \mathbf{R}'_r \mathbf{u}'_r$$

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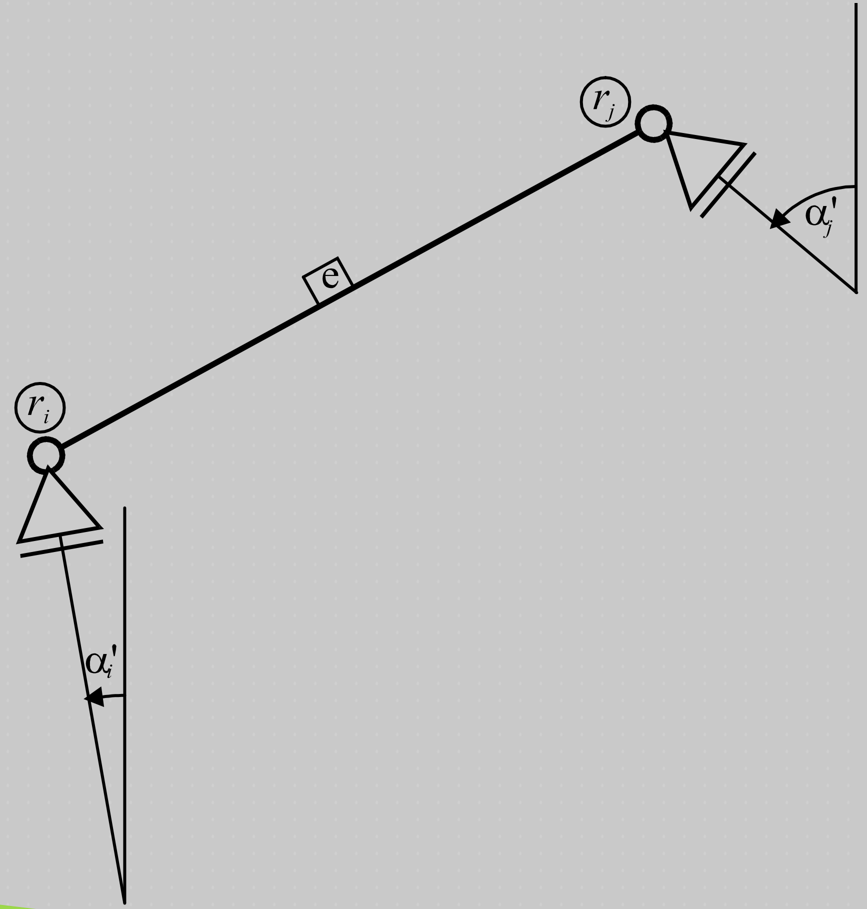
# STIFFNESS MATRIX FOR A 'SKEW' SUPPORT

- In equation $\mathbf{f}'_r = \left(\mathbf{R}'_r\right)^{\mathrm{T}} \mathbf{f}_r$ and $\mathbf{u}_r = \mathbf{R}'_r \mathbf{u}'_r$ we have marked:

$$\mathbf{R}'_r = \begin{bmatrix} c' & -s' \\ s' & c' \end{bmatrix}$$

$$c' = \cos\alpha'$$

$$s' = \sin\alpha'$$

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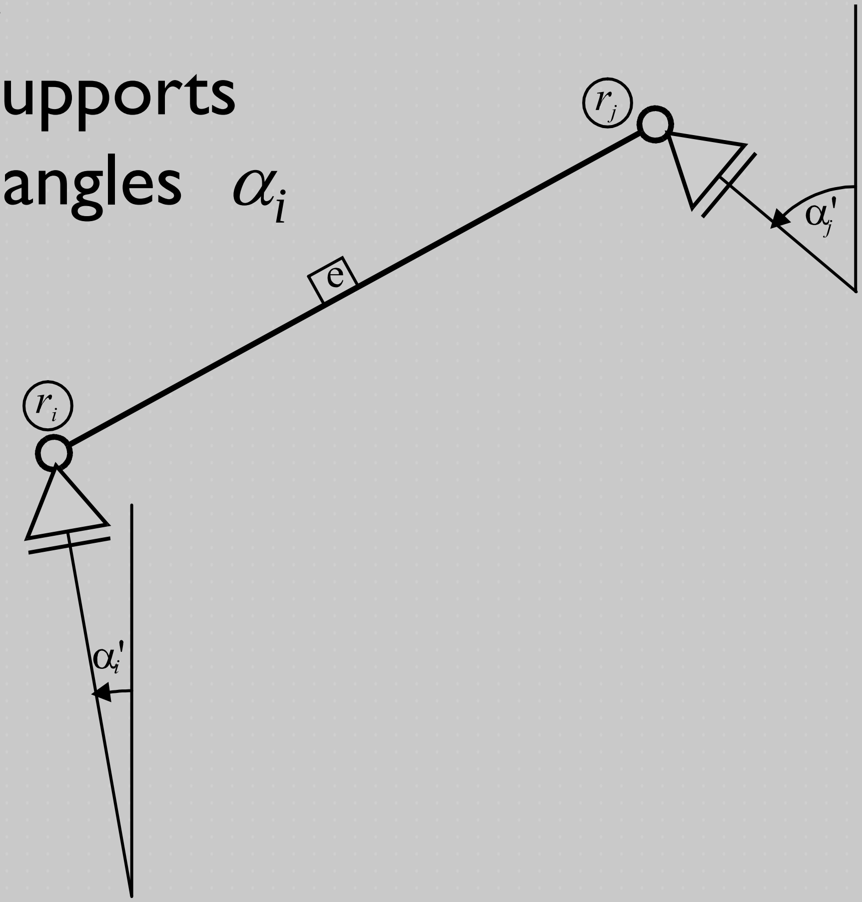
# STIFFNESS MATRIX FOR A 'SKEW' SUPPORT

- Let us assume that an element $e$ joins nodes $r_i$ and $r_j$ supported by 'skew' supports which are rotated by angles $\alpha_i$ and $\alpha_j$

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# STIFFNESS MATRIX FOR A 'SKEW' SUPPORT

- Then we write equilibrium equations for nodes $r_i$ and $r_j$ in the local coordinate system at the node $r_i$ and at the node $r_j$. The transformation of nodal forces vectors and nodal displacements vectors of the element $e$ is as follows: $\mathbf{f}'^e = \left(\mathbf{R}'^e\right)^{\mathsf{T}} \mathbf{f}^e$ for a nodal forces vector,

or in a developed form
$$\begin{bmatrix} \mathbf{f}'_{r_i} \\ \mathbf{f}'_{r_j} \end{bmatrix} = \begin{bmatrix} \left(\mathbf{R}'_{r_i}\right)^{\mathsf{T}} & \mathbf{0} \\ \mathbf{0} & \left(\mathbf{R}'_{r_j}\right)^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} \mathbf{f}_{r_i} \\ \mathbf{f}_{r_j} \end{bmatrix}$$

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# STIFFNESS MATRIX FOR A 'SKEW' SUPPORT

- for the nodal displacements vector:

$$\mathbf{u}^{e} = \mathbf{R}'^{e}\, \mathbf{u}'^{e}$$

or

$$\begin{bmatrix} \mathbf{u}_{r_i} \\ \mathbf{u}_{r_j} \end{bmatrix} = \begin{bmatrix} \mathbf{R}'_{r_i} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}'_{r_j} \end{bmatrix} \begin{bmatrix} \mathbf{u}'_{r_i} \\ \mathbf{u}'_{r_j} \end{bmatrix}$$

# STIFFNESS MATRIX FOR A ‘SKEW’ SUPPORT

- Inserting relationship $\mathbf{u}^e = \mathbf{R}'^e \mathbf{u}'^e$ into $\mathbf{f}^e = \mathbf{K}^e \mathbf{u}^e$ and the result into $\mathbf{f}'^e = \left(\mathbf{R}'^e\right)^\mathsf{T} \mathbf{f}^e$, we get the equation transforming the stiffness matrix of the element e from the global coordinate system to the support coordinate system:

$$\mathbf{f}'^e = \left(\mathbf{R}'^e\right)^\mathsf{T} \mathbf{K}^e \mathbf{R}'^e \mathbf{u}'^e$$

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# STIFFNESS MATRIX FOR A 'SKEW' SUPPORT

$$\mathbf{f}'^{e} = \left(\mathbf{R}'^{e}\right)^{\mathrm{T}} \mathbf{K}^{e} \mathbf{R}'^{e} \, \mathbf{u}'^{e}$$

- We simplify this equation to the form:

$$\mathbf{f}'^{e} = \mathbf{K}'^{e} \, \mathbf{u}'^{e}$$

in which we make use of the substitution:

$$\mathbf{K}'^{e} = \left(\mathbf{R}'^{e}\right)^{\mathrm{T}} \mathbf{K}^{e} \mathbf{R}'^{e}$$

defining the element matrix in the support coordinate system.

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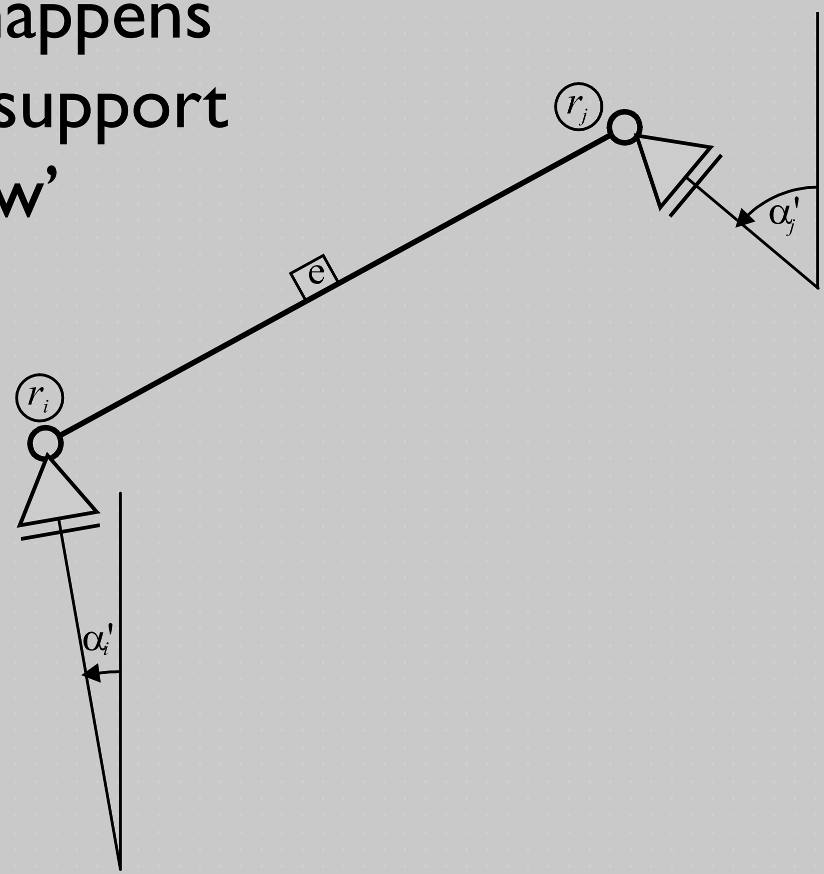
# STIFFNESS MATRIX FOR A 'SKEW' SUPPORT

- One of angles $\alpha'$ is most often equal to zero because it rarely happens that a truss bar joins two support nodes supported by a 'skew' support.
- The transformation matrix of a zero angle is a unit matrix.

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# STIFFNESS MATRIX FOR A 'SKEW' SUPPORT

- Because ($c'$=1, $s'$=0), then when the second node is described in the global system but we transform forces and displacements at the first node $r_i$, the element transformation matrix is simplified to the form:

$$\mathbf{R}'^{e} = \begin{bmatrix} \mathbf{R}'_{r_i} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

and when the transformation concerns the last node $r_j$ only:

$$\mathbf{R}'^{e} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}'_{r_j} \end{bmatrix}$$

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# STIFFNESS MATRIX FOR A 'SKEW' SUPPORT

- As it has been shown that the existence of 'skew' supports complicates the simple FEM algorithm presented in previous presentation because it requires additional transformations of stiffness matrices before the aggregation of the global matrix is done. There are other simpler, though approximate, methods of solving this problem and they will be discussed in the next section.

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# ELASTIC SUPPORTS AND BOUNDARY ELEMENTS

- Not all kinds of supports applied to support trusses can be described by the boundary conditions of types $\mathbf{u}_{rX} = 0$, $\mathbf{u}_{rY} = 0$ and $\mathbf{u}_{ry'} = 0$.
- There are flexible supports which have displacements connected with a support reaction, for instance, the linear relation of the following type:

$$R_{rX} = -h_{rX} u_{rX} \qquad R_{rY} = -h_{rY} u_{rY}$$

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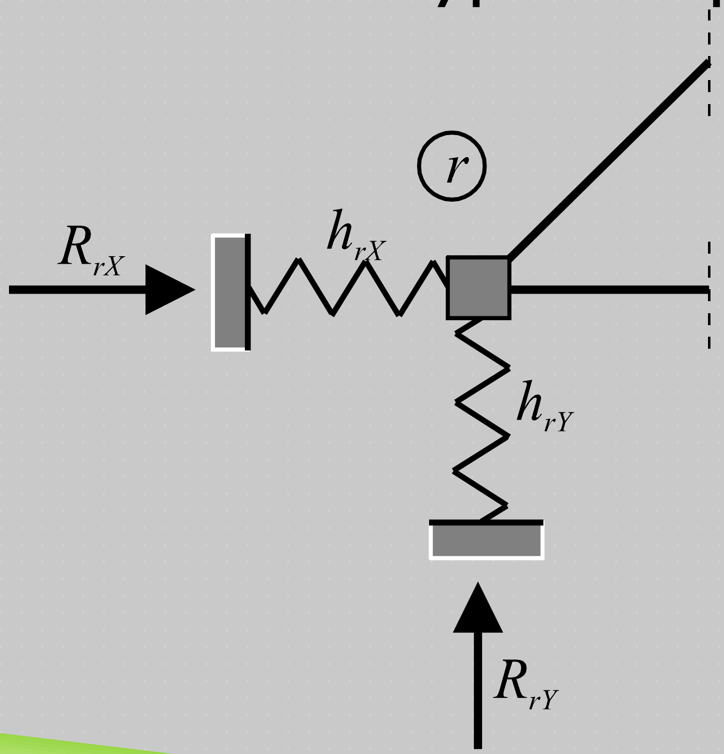
# ELASTIC SUPPORTS AND BOUNDARY ELEMENTS

- $h_{rX}$ is the support stiffness in the direction of the $X$ axis,
- $h_{rY}$ is the support stiffness in the direction of the $Y$ axis.
- The linear spring is a good model of this type of support.

$$R_{rX} = -h_{rX} u_{rX}$$

$$R_{rY} = -h_{rY} u_{rY}$$



# ELASTIC SUPPORTS AND BOUNDARY ELEMENTS

- If we treat reactions $R_{rX}$ and $R_{rY}$ acting on the node supported elastically as external forces, then we obtain the nodal forces vector containing unknown displacements $u_{rX}$, $u_{rY}$:

$$p = \left[\begin{array}{c} P_{1X} \\ P_{1Y} \\ \hline P_{2X} \\ P_{2Y} \\ \hline \vdots \\ \hline R_{rX} \\ R_{rY} \\ \hline \vdots \\ \hline P_{N_nX} \\ P_{N_nY} \end{array}\right] \begin{array}{c} 1 \\ \\ 2 \\ \\ \vdots \\ r \\ \\ \vdots \\ N_n \\ \\ \end{array} = \left[\begin{array}{c} P_{1X} \\ P_{1Y} \\ P_{2X} \\ P_{2Y} \\ \vdots \\ -h_{rX}u_{rX} \\ -h_{rY}u_{rY} \\ \vdots \\ P_{N_nX} \\ P_{N_nY} \end{array}\right]$$

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# ELASTIC SUPPORTS AND BOUNDARY ELEMENTS

- The vector $\mathbf{p}$ cannot be absolutely used as the right side of $\mathbf{Ku}=\mathbf{p}$ in which unknown values of nodal displacements should be on the left side of the equation. Now we are transforming the vector $\mathbf{p}$ described in previous slide in such a way that nodal reactions of the elastic node *r* will be moved to the left side of the equilibrium equation:

$$\mathbf{K}^s\,\mathbf{u} = \mathbf{p}^r$$

$$\mathbf{K}^s\mathbf{u} = \mathbf{p}^r$$

- where

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# ELASTIC SUPPORTS AND BOUNDARY ELEMENTS

- $\mathbf{K}^s$- the stiffness matrix containing information about elastic supports of the structure,
- $\mathbf{p}^r$- the nodal forces vector in which the boundary conditions written in equation $\mathbf{K}^0\,\mathbf{u}^s = \mathbf{p}^r$
- we can treat the elastic supports as fixed ones after transferring the relations which described them to the left hand side of the equation) are considered.

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# ELASTIC SUPPORTS AND BOUNDARY ELEMENTS

- The matrix $\mathbf{K}^s$ is written by the equation:

$$
\mathbf{K}^s = \begin{array}{cc}
\left[\begin{array}{cc:c:cc:c:c}
\mathbf{K}_{11} & \mathbf{K}_{12} & & \mathbf{K}_{1m} & \mathbf{K}_{1(m+1)} & & \mathbf{K}_{1N_n} \\
\mathbf{K}_{21} & \mathbf{K}_{22} & \cdots & \mathbf{K}_{2m} & \mathbf{K}_{2(m+1)} & \cdots & \mathbf{K}_{2N_n} \\
\hdashline
\vdots & & \ddots & \vdots & & & \vdots \\
\hdashline
\mathbf{K}_{m1} & \mathbf{K}_{m2} & & \mathbf{K}_{mm} + h_{rX} & \mathbf{K}_{m(m+1)} & & \mathbf{K}_{mN_n} \\
\mathbf{K}_{(m+1)1} & \mathbf{K}_{(m+1)2} & \cdots & \mathbf{K}_{(m+1)m} & \mathbf{K}_{(m+1)(m+1)} + h_{rY} & \cdots & \mathbf{K}_{(m+1)N_n} \\
\hdashline
\vdots & & & \vdots & & \ddots & \vdots \\
\hdashline
\mathbf{K}_{N_n1} & \mathbf{K}_{N_n2} & \cdots & \mathbf{K}_{N_nm} & \mathbf{K}_{N_n(m+1)} & \cdots & \mathbf{K}_{N_nN_n}
\end{array}\right]
&
\begin{array}{c}
1 \\ \\ \vdots \\ r \\ \\ \mathrm{N_n} \\ \\
\end{array}
\end{array}
$$

# ELASTIC SUPPORTS AND BOUNDARY ELEMENTS

- $m$ - the global number of the first degree of freedom of the node $r$. With standard numbering $m=(r-1)N_D+1$ where $N_D$ is the number of degrees of freedom of the node.
- For a 2D truss $N_D=2$, the number of the first degree of freedom of the node $r$ is equal to $m=2r-1$.

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# ELASTIC SUPPORTS AND BOUNDARY ELEMENTS

- At this stage, the modified matrix $\mathbf{K}^s$ contains the stiffness of elastic supports which are added to the terms coming from the truss element of a structure. These sums are located on the main diagonal of the matrix in rows describing the equilibrium of the node $r$. Such an interpretation of elastic supports leads to a convenient, although simplistic, way of considering fixed supports.
- We substitute them for elastic supports with very large stiffness, for example $H=1\times10^{30}$ onto the main diagonal.

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# ELASTIC SUPPORTS AND BOUNDARY ELEMENTS

- This method was formulated by Irons (1980) who multiplies terms lying in a suitable row on the diagonal of the matrix $\mathbf{K}$ by numbers of the order of $10^6$. After inserting a high value onto the diagonal, it is irrelevant to insert zeros in the matrix $\mathbf{K}$ and the vector $\mathbf{p}$.
- It is very important for large stiffness matrices which are often stored in structures of data different from quadratic tables. The simplicity of this method ensures that it is commonly used in the computer implementation of the FEM algorithm instead of the exact method described previously in this presentation.

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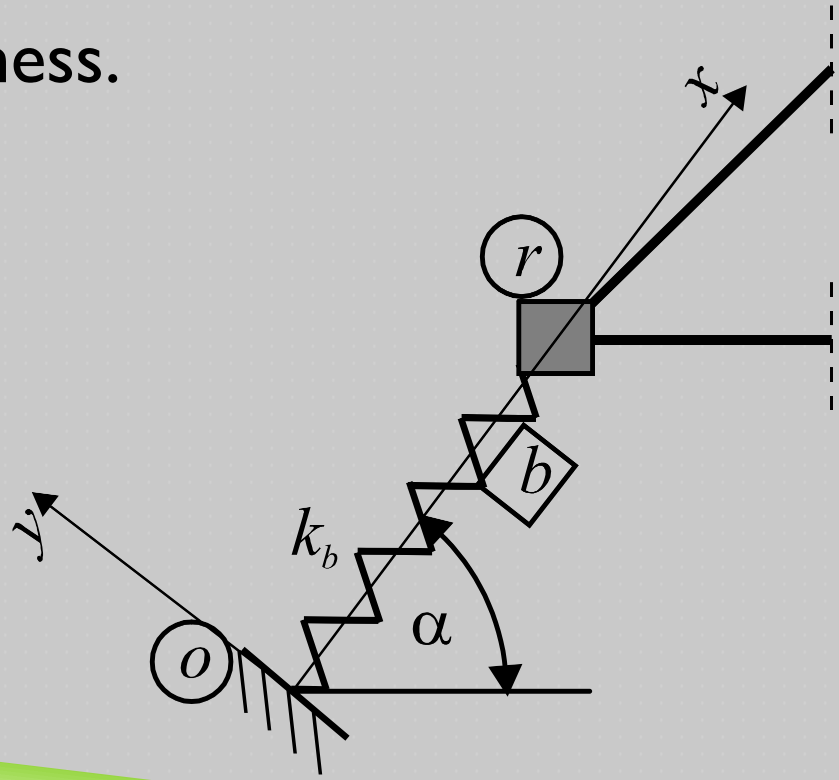
# ELASTIC SUPPORTS AND BOUNDARY ELEMENTS

- Elastic supports also suggest the use of an element which could substitute any elastic constraints and fixed supports which should be treated as elastic supports with large stiffness.

- This is a support element rotated by an angle α with respect to the global coordinates.

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# ELASTIC SUPPORTS AND BOUNDARY ELEMENTS

- We can easily obtain the stiffness matrix of such an element from the matrix of an ordinary truss element described by $\mathbf{K}'^e$ in the local coordinate system or $\mathbf{K}^e$ in the global system. We do it in such a way that we substitute the stiffness of a bar $EA/L$ for the stiffness of the elastic boundary element $k_b$.

$$\mathbf{K}^e = \begin{bmatrix} \mathbf{J} & -\mathbf{J} \\ -\mathbf{J} & \mathbf{J} \end{bmatrix}$$

$$\mathbf{K}'^e = \begin{bmatrix} \frac{EA}{L} & 0 & -\frac{EA}{L} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{EA}{L} & 0 & \frac{EA}{L} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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# ELASTIC SUPPORTS AND BOUNDARY ELEMENTS

- In general, the node *o* of this element is always fixed, so we can remove it from the set of equations which allows us to treat the boundary element as an element with two degrees of freedom:

$$\mathbf{K}^b = k_b \begin{bmatrix} c^2 & sc \\ sc & s^2 \end{bmatrix}$$

where, as before

$$c = \cos\alpha, \quad s = \sin\alpha.$$

- When we want to substitute the fixed support for this element we accept $k_b = H$. The value of $H$ depends on the computer system in which the program will be started and most of all it depends on the type of real numbers. We can take for example $H = 1 \times 10^{30}$ as reference for many systems.

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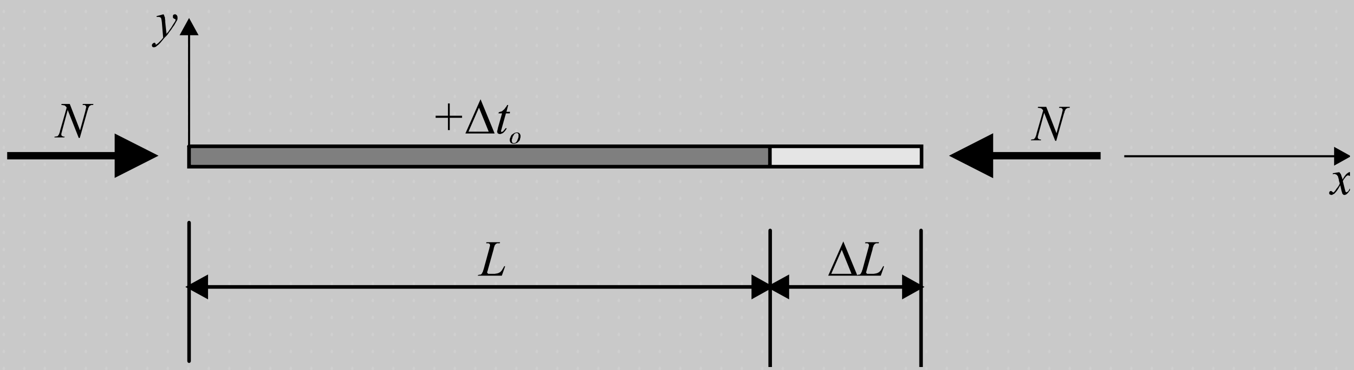
# THE NODAL LOADS VECTOR WITH TEMPERATURE LOAD

- As we have already noted in the introduction to this Chapter, truss loads which act on elements and do not act on nodes directly are temperature loads.
- Now we will show how we can replace this load by known loads, that is, concentrated forces acting on the nodes of a structure.

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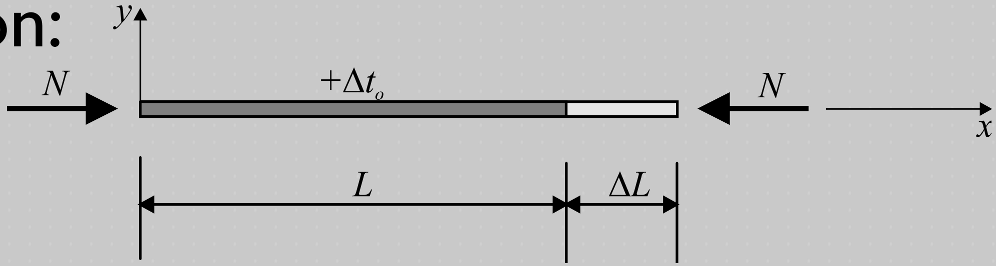
# THE NODAL LOADS VECTOR WITH TEMPERATURE LOAD

- As we know, the increase in temperature of an element causes it to lengthen which, with the assumption of a steady increase in the temperature of the whole bar, is described by the equation:




$$\varepsilon_t = \frac{\Delta L}{L} = \alpha_t \Delta t_o$$

- $\alpha_t$ - the coefficient of thermal expansion of the material from which the element is made,
- $\Delta t_o$ - stands for an increment of temperature in the middle fibres (joining centres of gravity of cross sections of an element).

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# THE NODAL LOADS VECTOR WITH TEMPERATURE LOAD

- We assume a steady increase in temperature in the whole section and homogeneity of the material. The element has no freedom to grow but is limited by fixed nodes, and we obtain an axial force which is set up within the element:

$$N = -\int_{\mathscr{A}} \sigma_t \, d\mathscr{A} = -\int_{\mathscr{A}} E\varepsilon_t \, d\mathscr{A} = -\int_{\mathscr{A}} E\alpha_t \Delta t_o \, d\mathscr{A} = -E\alpha_t \Delta t_o A$$

$E$ - Young's modulus of the material,

$\mathscr{A}$ - the surface area of the cross section.

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# THE NODAL LOADS VECTOR WITH TEMPERATURE LOAD

- The nodal forces vector of the element due to the temperature, written in the local coordinate system *xy*, is equal to:

$$\mathbf{f}'^{et} = EA\alpha_t \Delta t_o \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

after transformation to the global system:

$$\mathbf{f}^{et} = EA\alpha_t \Delta t_o \begin{bmatrix} c \\ s \\ -c \\ -s \end{bmatrix}$$

where $c = \cos\alpha$, $s = \sin\alpha$.

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# THE NODAL LOADS VECTOR WITH TEMPERATURE LOAD

- Since forces acting on the nodes are necessary for the equilibrium equations, and as it is known, they are of opposite direction to other forces acting on elements, then we subtract them from other forces while building the global nodal forces vector.

  This is shown in next slide.

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# THE NODAL LOADS VECTOR WITH TEMPERATURE LOAD

$n_i$ – global node of the first node of an element

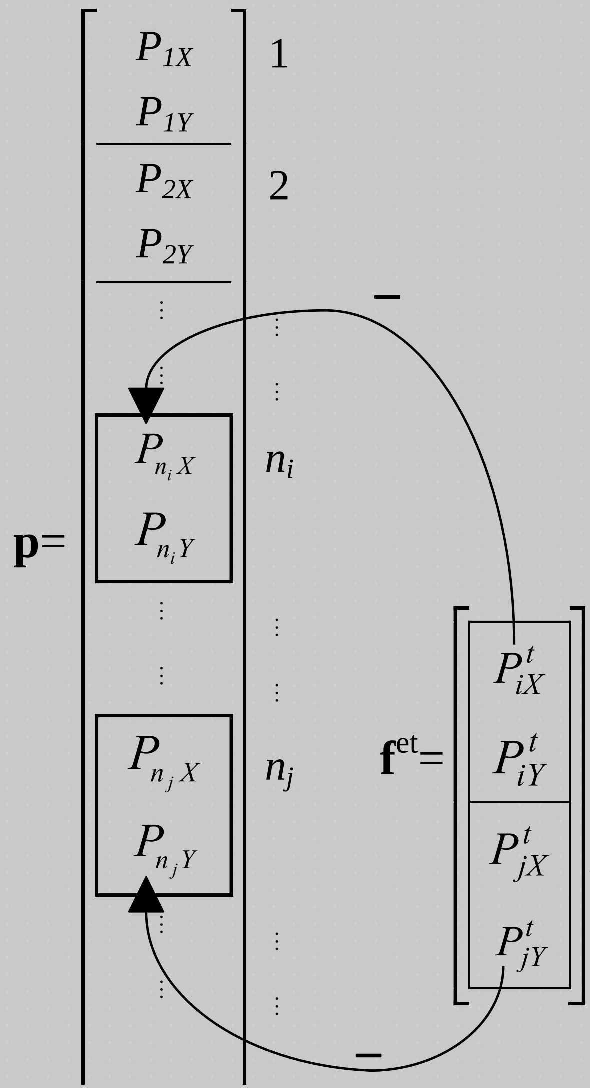
$n_j$ – global node of the last node of an element

$$P^t_{iX} = EA\alpha_t \Delta t_o \cos\alpha$$

$$P^t_{iY} = EA\alpha_t \Delta t_o \sin\alpha$$

$$P^t_{jX} = -EA\alpha_t \Delta t_o \cos\alpha$$

$$P^t_{jY} = -EA\alpha_t \Delta t_o \sin\alpha$$

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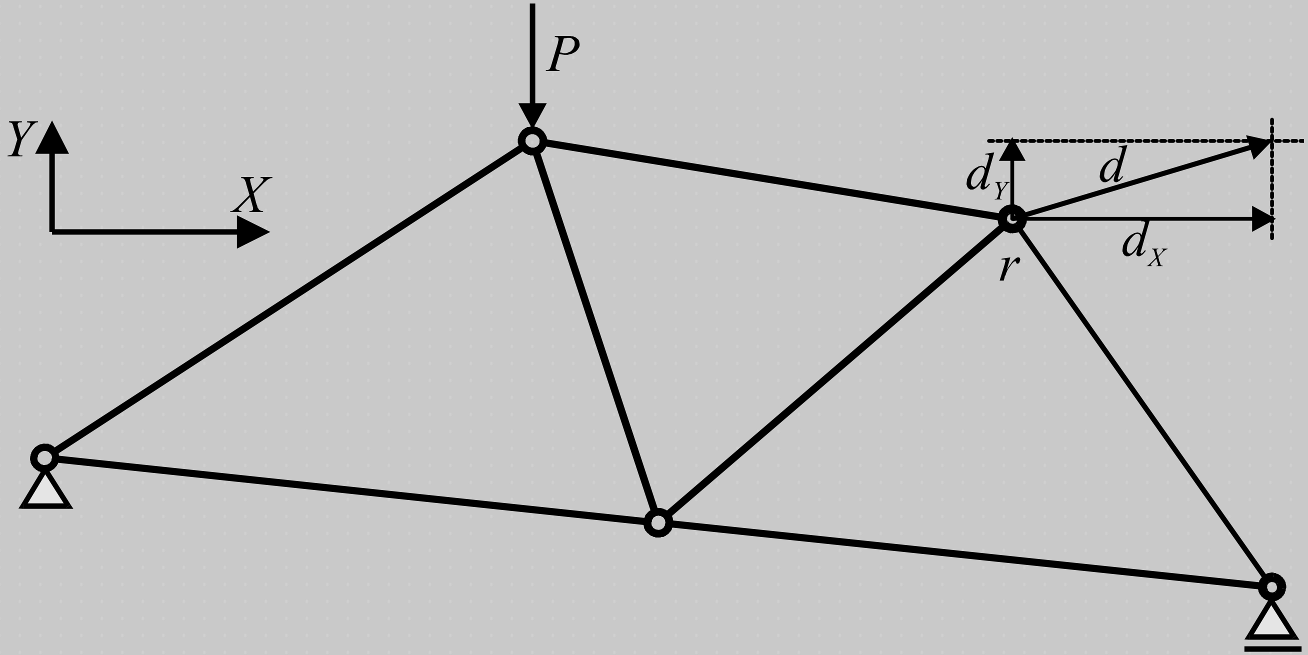
# THE GEOMETRIC LOAD ON A TRUSS

- The final type of truss load, which we will describe, is the geometric load – forced displacements of nodes.

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# THE GEOMETRIC LOAD ON A TRUSS

- We assume that the node *r* is displaced by the vector $\mathbf{d}$. It is necessary to apply forces to the node to cause this displacement. Values of these forces are not known, whereas we know components of the displacement of the node $r$ : (*) $u_{rX} = d_X\,,\, u_{rY} = d_Y\,,$ where $d_X, d_Y$ are the components of the vector of the forced displacement $\mathbf{d}$.
- Equation (*) is like the known equations of the boundary conditions $\mathbf{u}_{rX} = \mathbf{0}$ and $\mathbf{u}_{rY} = \mathbf{0}$ but with one difference, here we have obtained nonhomogeneous equations. It changes the procedure of symmetrisation of the stiffness matrix.

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# THE GEOMETRIC LOAD ON A TRUSS

- Previously we inserted zeros into suitable columns of the matrix $\mathbf{K}$ which did not induce any consequences. At this time we have to keep the components of the matrix occurring in this column because they are multiplied by given displacements ($u_{rX} = d_X$ , $u_{rY} = d_Y$) and they are usually not equal to zero.
- Hence transformations of $\mathbf{K}$ and $\mathbf{p}$ leading to the consideration of the geometric load should look as follows:

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# THE GEOMETRIC LOAD ON A TRUSS

- We form vectors $\mathbf{k}_{rX}$ and $\mathbf{k}_{rY}$ which are suitable columns of the matrix $\mathbf{K}$ joined with the displacements of the node $r$. Vector $\mathbf{k}_{rX}$ is the column with a number equal to the displacement global number $u_{rX}$ and $\mathbf{k}_{rY}$ - to the displacement global number $u_{rY}$.
- We move the nodal forces due to the known displacements $d_X$ and $d_Y$ to the right hand side of the set of equations:

$$\mathbf{p}^d = \mathbf{p} - \mathbf{k}_{rX} d_X - \mathbf{k}_{rY} d_Y$$

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# THE GEOMETRIC LOAD ON A TRUSS

- There is one difference in boundary conditions. We put known values into the rows of the right hand side vector These rows have the global numbers equivalent to the degrees of freedom $u_{rX}$ and $u_{rY}$.
- After making the above transformations, the following set of equations rises: $\mathbf{K}^r\,\mathbf{u} = \mathbf{p}^{rd}$

$\mathbf{K}^r$ - the stiffness matrix modified by the standard consideration of the boundary conditions,

$\mathbf{p}^{rd}$ - the modified vector determined by equation:

$\mathbf{p}^d = \mathbf{p} - \mathbf{k}_{rX}\,d_X - \mathbf{k}_{rY}\,d_Y$ after inserting known values of displacements.

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# THE GEOMETRIC LOAD ON A TRUSS

- These displacements are:

$$P_{rX} = d_X \quad P_{rY} = d_Y$$

$$\mathbf{p}^{d} = \left[\begin{array}{c} P_{1X}\\ P_{1Y}\\ \hline P_{2X}\\ P_{2Y}\\ \hline \vdots\\ \vdots\\ \hline P_{rX}\\ P_{rY}\\ \hline \vdots\\ \vdots\\ \hline \vdots\\ \vdots\\ \hline \vdots\\ \vdots\\ \hline P_{NnX}\\ P_{NnY} \end{array}\right]\begin{array}{c} 1\\ \\ 2\\ \\ \vdots\\ \vdots\\ n_r\\ \\ \vdots\\ \vdots\\ \vdots\\ \vdots\\ \vdots\\ \vdots\\ N_n\\ \\ \end{array} - \mathbf{k}_{rX}\cdot d_X - \mathbf{k}_{rY}\cdot d_Y \qquad \mathbf{p}^{rd} = \left[\begin{array}{c} \ \\ \ \\ \hline \ \\ \ \\ \hline \ \\ \ \\ \hline d_X\\ d_Y\\ \ \\ \ \\ \hline \ \\ \ \\ \hline \ \\ \ \\ \hline \ \\ \ \end{array}\right]\begin{array}{c} 1\\ \\ 2\\ \\ \vdots\\ \vdots\\ n_r\\ \\ \vdots\\ \vdots\\ \vdots\\ \vdots\\ \vdots\\ \vdots\\ N_n\\ \\ \end{array}$$

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# REACTIONS, INTERNAL FORCES AND STRESSES IN ELEMENTS

- After aggregation of the stiffness matrix, consideration of the boundary conditions and building the nodal forces vector, we obtain the set of linear equations in forms :

  $\mathbf{K}^r\,\mathbf{u} = \mathbf{p}^r\;,\mathbf{K}^s\,\mathbf{u} = \mathbf{p}^r\;,\mathbf{K}^r\,\mathbf{u} = \mathbf{p}^{rd}\;.$

  with a positively determined symmetric matrix.
- The solution of the set of equations is the nodal displacements vector of a structure.

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# REACTIONS, INTERNAL FORCES AND STRESSES IN ELEMENTS

- Knowing nodal displacements allows us to determine control sums of nodes and support reactions in the support nodes in a very simple way. And then we make use of equation $\mathbf{Ku}=\mathbf{p}$ in which the matrix $\mathbf{K}$ does not contain any information about the support constraints.

$$\mathbf{r} = \mathbf{K}\mathbf{u} - \mathbf{p}$$

- The vector of reactions $\mathbf{r}$ should contain zeros at free nodes and values of reactions at support nodes. If we assume the occurrence of the local coordinate system in some nodes (the 'skew' supports), then the components of reactions will be expressed in the local coordinate system.

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# REACTIONS, INTERNAL FORCES AND STRESSES IN ELEMENTS

- Since numerical errors resulting from approaching values of numbers stored in the computer memory increase during the solution process, the control sums are rarely equal to zero and they are most often small numbers, for example the order of $1 \times 10^{-10}$.
- Components of the global displacements vector enable the building of global displacements vectors for the elements.

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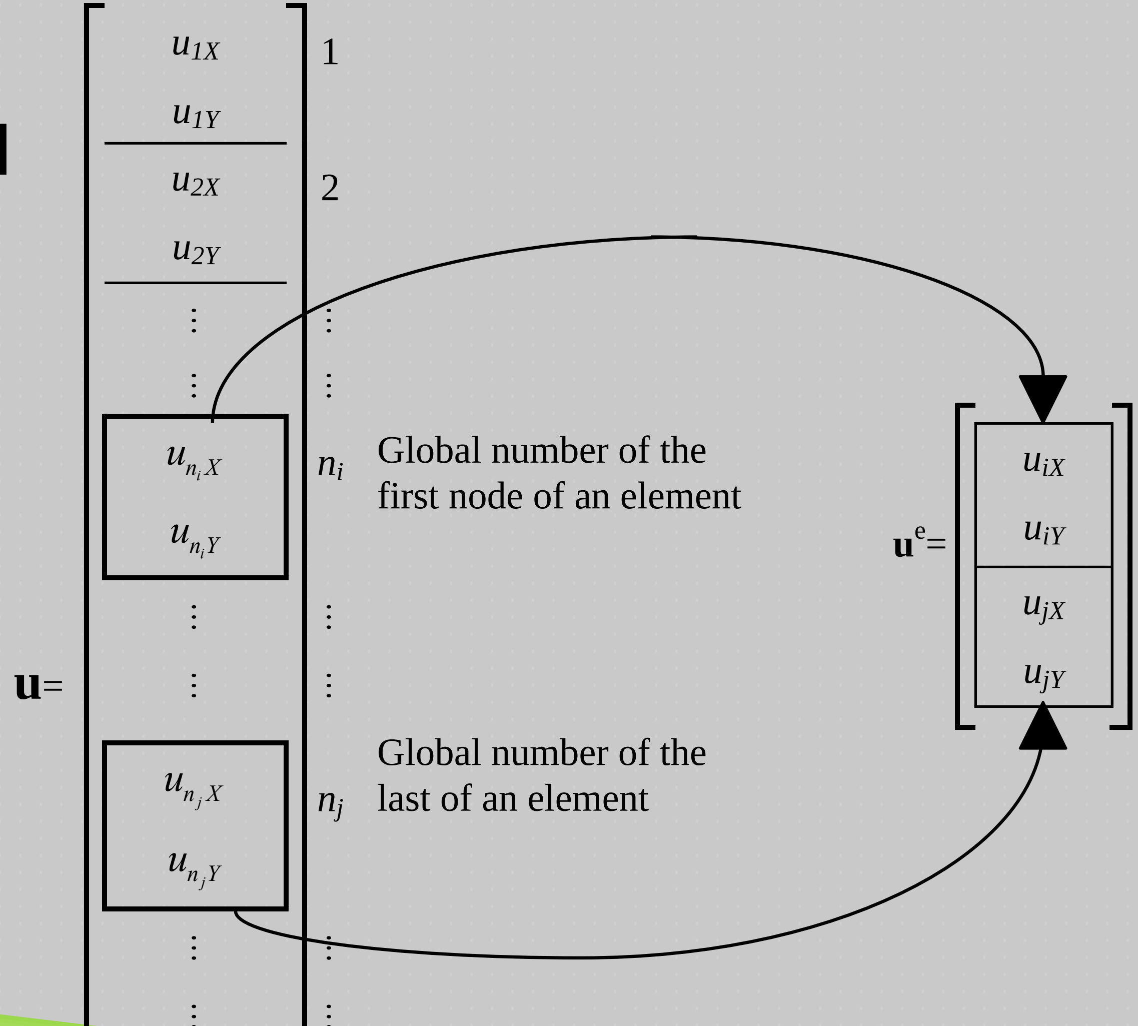
# REACTIONS, INTERNAL FORCES AND STRESSES IN ELEMENTS

The geometric load included into the global stiffness matrix.



# REACTIONS, INTERNAL FORCES AND STRESSES IN ELEMENTS

- Since the components of the vector $\mathbf{u}$ are not always written in the global coordinate system (the 'skew' supports), then it can happen that some components of $\mathbf{u}^e$ are expressed in the global system and others in local. We standardise the description of the vector bringing down the components to the global coordinate system by taking advantage of equation $\mathbf{u}^e = \mathbf{R}'^e\,\mathbf{u}'^e$.

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# REACTIONS, INTERNAL FORCES AND STRESSES IN ELEMENTS

- It should be noted that the standarisation is only necessary for elements joined to a node which is supported by a skew support.
- Nodal displacements of an element allow the internal force $N$ in a truss element to be calculated quite easily.

# REACTIONS, INTERNAL FORCES AND STRESSES IN ELEMENTS

- We can either make use of $N = \frac{EA}{L}\left(u_{jx} - u_{ix}\right)$ which requires knowledge of displacements in the local coordinate system of the element or on the basis of Eqn. $F_{ix} = -N$, $F_{jx} = N$, $\mathbf{K}'^{e}\,\mathbf{u}'^{e} = \mathbf{f}'^{e}$ and $\mathbf{f}'^{e} = (\mathbf{R}^{e})^{\mathsf{T}}\,\mathbf{f}'^{e}$ we search the relationship:

$$N = \frac{EA}{L}\left[c\left(u_{jX} - u_{iX}\right) + s\left(u_{jY} - u_{iY}\right)\right]$$

where $c = \cos\alpha$, $s = \sin\alpha$.

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# REACTIONS, INTERNAL FORCES AND STRESSES IN ELEMENTS

- Stresses in the truss element, assuming that the bar is homogeneous, are the axial stresses only which can be calculated using a simple relationship:

$$\sigma_x = \frac{N}{A} = \frac{E}{L}\left[c\left(u_{jX} - u_{iX}\right) + s\left(u_{jY} - u_{iY}\right)\right]$$

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# REACTIONS, INTERNAL FORCES AND STRESSES IN ELEMENTS

- If the element is loaded with a temperature gradient, then the correction coming from thermal expansion of the material should be taken into consideration:

$$\sigma_x = E(\varepsilon - \varepsilon_t) = \frac{E}{L}\left[c\left(u_{jX} - u_{iX}\right) + s\left(u_{jY} - u_{iY}\right) - L(\alpha_t \Delta t_o)\right]$$

$$N = A\sigma_x = \frac{EA}{L}\left[c\left(u_{jX} - u_{iX}\right) + s\left(u_{jY} - u_{iY}\right) - L(\alpha_t \Delta t_o)\right]$$

- These calculation completes the static analysis of the truss.

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