FINITE ELEMENT METHOD
Brick elements

Jerzy Podgórski

Introduction

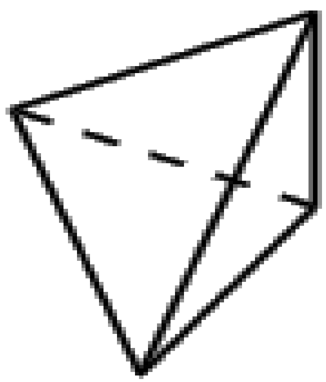
Brick is the Three-dimensional element, which can be defined as a body, to which all dimensions are of the same order. The shape of the body and the load is any of available. With brick elements, fully 3D solid constructions can be modeled.

Introduction

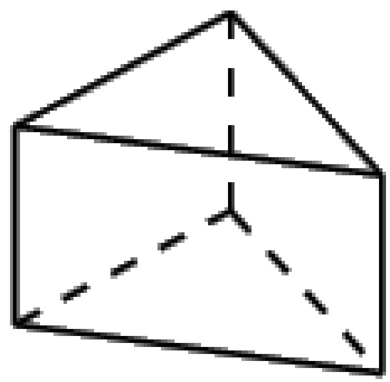
Brick elements can replace every other type of element, like frame, shell and plate elements. A solid model is divided into brick elements and the geometric shape of the elements can be tetrahedron, hexahedron or prism with triangle base, which means, that a brick element is build of triangles and quadrilateral

Introduction

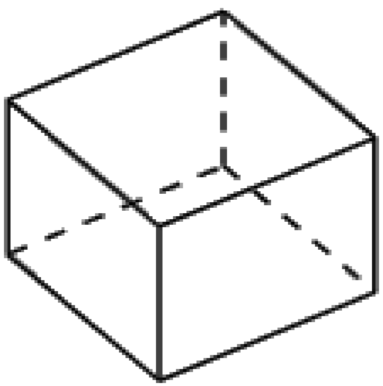
Three-dimensional shapes of brick elements



four-nodes
(tetrahedron)



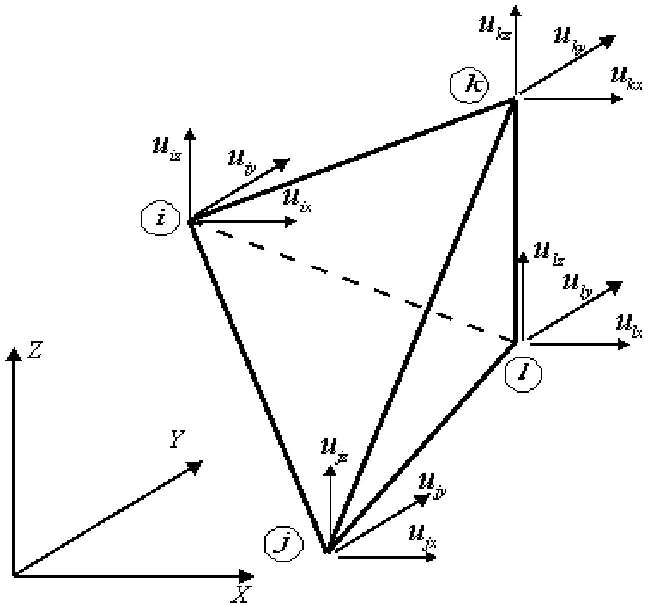
six-nodes

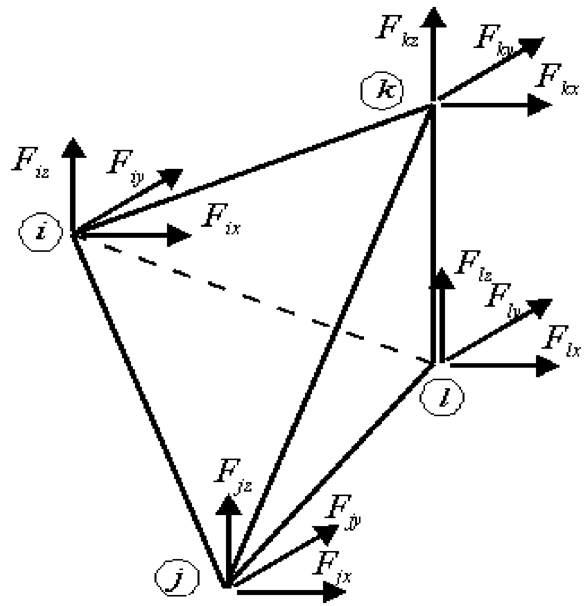


eight-nodes
(hexahedron)

Introduction

We will show how the tetrahedron is regarded in Finite Element Method. You can see how displacements and nodal forces are located





Introduction

As it can be seen, in each node there are three displacements and forces in all global directions, but there are no rotations and moments. Vector of movements of element nodes and nodal forces can be written as follows

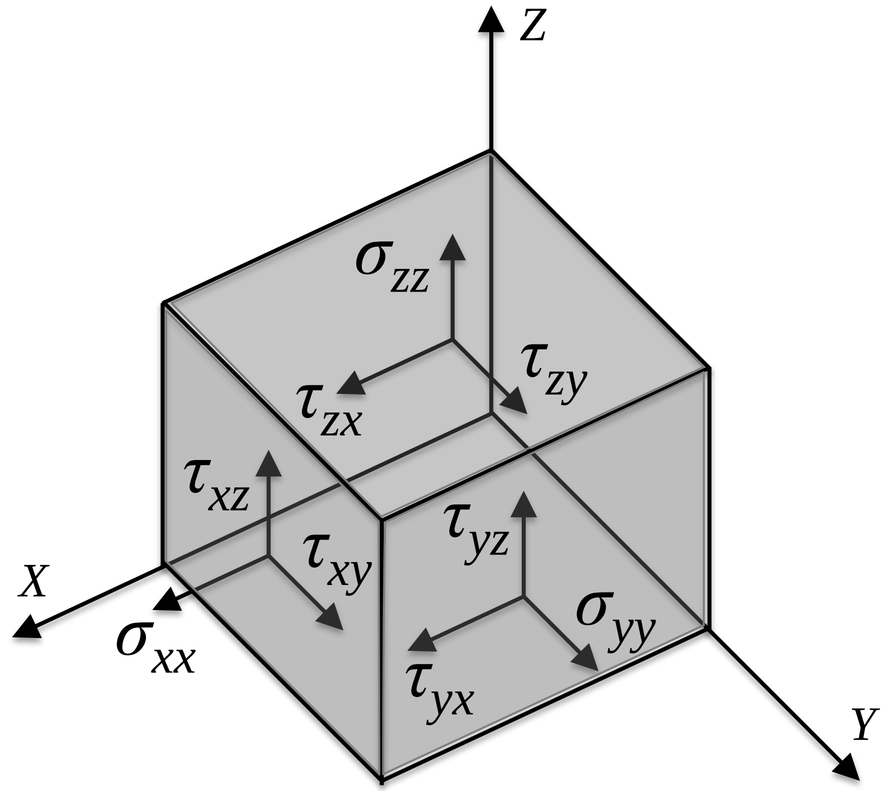
Introduction

Vector of movements of element nodes and nodal forces

$$\mathbf{u}'^{\mathbf{e}} = \begin{bmatrix} u_{ix} \\ u_{iy} \\ u_{iz} \\ u_{jx} \\ u_{jy} \\ u_{jz} \\ u_{kx} \\ u_{ky} \\ u_{kz} \\ u_{lx} \\ u_{ly} \\ u_{lz} \end{bmatrix} \qquad \mathbf{f}'^{\mathbf{e}} = \begin{bmatrix} F_{ix} \\ F_{iy} \\ F_{iz} \\ F_{jx} \\ F_{jy} \\ F_{jz} \\ F_{kx} \\ F_{ky} \\ F_{kz} \\ F_{lx} \\ F_{ly} \\ F_{lz} \end{bmatrix}$$

Relation between strain, stress and displacements

The brick works in a spatial state of stress



Relation between strain, stress and displacements

In brick element, there are three components of normal stresses σ_{xx}, σ_{yy}, σ_{zz}, and six sheer stresses, but according to the symmetry of a stress tensor components of shear stress we have $\tau_{xy}=\tau_{yx}$, $\tau_{xz}=\tau_{zx}$, and $\tau_{zy}=\tau_{yz}$, thus we have six independent components of stress which are composed in the stress vector

Relation between strain, stress and displacements

components of the stress vector

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix}$$

Relation between strain, stress and displacements

Because of the fact that the brick works in three-dimensional state of stress and strain, the strain vector is similar to stress vector. Relationships between the components of displacement and strain vectors are similar to those for 2D elements

Relation between strain, stress and displacements

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u_x}{\partial x} \\ \dfrac{\partial u_y}{\partial y} \\ \dfrac{\partial u_z}{\partial z} \\ \dfrac{\partial u_x}{\partial y} + \dfrac{\partial u_y}{\partial x} \\ \dfrac{\partial u_y}{\partial z} + \dfrac{\partial u_z}{\partial y} \\ \dfrac{\partial u_z}{\partial x} + \dfrac{\partial u_x}{\partial z} \end{bmatrix}$$

Relation between strain, stress and displacements

The relationship between vectors $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ is, like in two-dimensional elements, described by constitutive equations. For elastic isotropic materials the constitutive equation is shown below

$$\boldsymbol{\sigma} = \mathbf{D} \cdot \boldsymbol{\varepsilon}$$

where $\mathbf{D}$ is the square matrix with dimensions 6x6 containing the material constants

Stiffness matrix of 3D element

The stiffness matrix is the same as for all previous chapters

$$\mathbf{f}^e = \mathbf{K}^e \mathbf{u}^e$$

where $\mathbf{K}^e$ has the same formula as in previous chapters

Stiffness matrix of 3D element

$$\mathbf{K}^e = \int_V \left(\mathbf{B}^e\right)^{\mathrm{T}} \mathbf{D}\,\mathbf{B}^e dV$$

In case of brick element $\mathbf{B}^e$ has following formula

$$\mathbf{B}^e = \Re\mathbf{N}$$

where $\mathbf{N}$ is matrix of shape function (which will be described later), $\Re$ is a matrix of differential operators

Stiffness matrix of 3D element

$$\Re = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix}$$

Shape function of 3D elements

Displacement of every point of brick element in three-dimensional coordinate system can be written as a function

$$u_x(x,y,z) = a_1 + a_2 x + a_3 y + a_4 z$$

$$u_y(x,y,z) = b_1 + b_2 x + b_3 y + b_4 z$$

$$u_z(x,y,z) = c_1 + c_2 x + c_3 y + c_4 z$$

Shape function of 3D elements

These are displacements in every of three dimensions located in any point with coordinates (x, y, z). All functions are linear.

Shape function of 3D elements

Equation

$$u_x(x,y,z) = a_1 + a_2 x + a_3 y + a_4 z$$
$$u_y(x,y,z) = b_1 + b_2 x + b_3 y + b_4 z$$
$$u_z(x,y,z) = c_1 + c_2 x + c_3 y + c_4 z$$

can be written as a function depending on displacements in every of four nodes in tetrahedral element

Shape function of 3D elements

$$u_x(x,y,z) = N_i(x,y,z)u_{ix} + N_j(x,y,z)u_{jx} + N_k(x,y,z)u_{kx} + N_l(x,y,z)u_{lx}$$

$$u_y(x,y,z) = N_i(x,y,z)u_{iy} + N_j(x,y,z)u_{jy} + N_k(x,y,z)u_{ky} + N_l(x,y,z)u_{ly}$$

$$u_z(x,y,z) = N_i(x,y,z)u_{iz} + N_j(x,y,z)u_{jz} + N_k(x,y,z)u_{kz} + N_l(x,y,z)u_{lz}$$

where

$$N_i(x,y,z) = a_i + b_i x + c_i y + d_i z$$

$$N_j(x,y,z) = a_j + b_j x + c_j y + d_j z$$

$$N_k(x,y,z) = a_k + b_k x + c_k y + d_k z$$

$$N_l(x,y,z) = a_l + b_l x + c_l y + d_l z$$

Shape function of 3D elements

And this equation can be written in matrix form

$$\mathbf{u}(x, y, z) = \mathbf{N}\mathbf{u}'^{e}$$

where $\mathbf{u}(x, y, z)$ is a displacement vector of any point located inside the brick element, $\mathbf{u}'^{e}$ is nodal displacement vector, and $\mathbf{N}$ is stiffness function matrix

Shape function of 3D elements

$$\mathbf{u}(x,y,z) = \begin{bmatrix} u_x(x,y,z) \\ u_y(x,y,z) \\ u_z(x,y,z) \end{bmatrix}$$

$$\mathbf{N} = \begin{bmatrix} N_i & 0 & 0 & N_j & 0 & 0 & N_k & 0 & 0 & N_l & 0 & 0 \\ 0 & N_i & 0 & 0 & N_j & 0 & 0 & N_k & 0 & 0 & N_l & 0 \\ 0 & 0 & N_i & 0 & 0 & N_j & 0 & 0 & N_k & 0 & 0 & N_l \end{bmatrix}$$

Shape function of 3D elements

Shape function for element of eight nodes can be written for any node as bellow

$$N_i = \frac{1}{8}(1+\xi_o)(1+\eta_o)(1+\zeta_o)$$

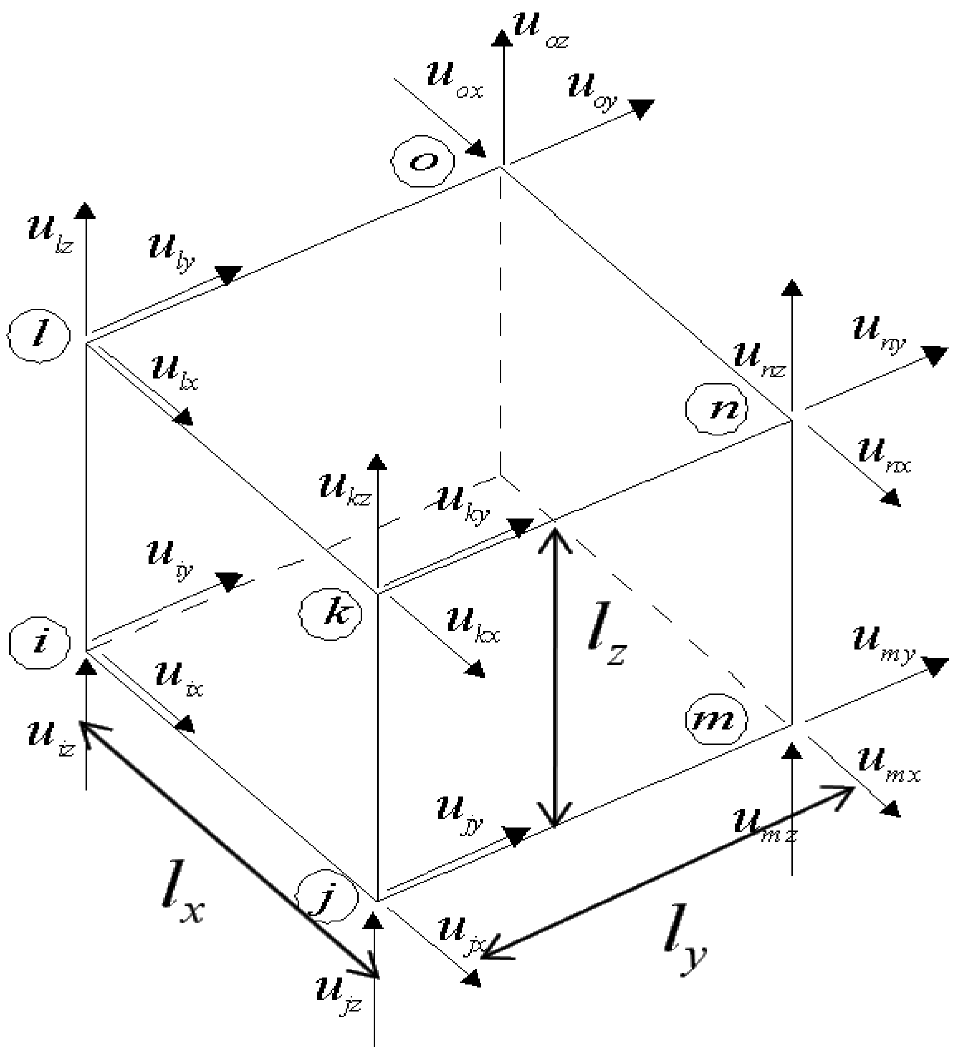
where

$$\xi_o = \pm\frac{x}{l_x} \qquad \eta_o = \pm\frac{y}{l_y} \qquad \zeta_o = \pm\frac{z}{l_z}$$

Shape function of 3D elements

This shape function is based on Lagrangian interpolation for the three variables of function passing through two points. Designations for eight-node element are shown in next Figure

Shape function of 3D elements



Shape function of 3D elements

Now, when we have the shape function, we can return equation ($\mathbf{B}^e = \Re \mathbf{N}$), where the geometric matrix for nodes $\mathbf{B}^e$ can be expressed for four-node element

Shape function of 3D elements

$$\mathbf{B}^e = \Re\mathbf{N} = \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 & 0 & \frac{\partial N_j}{\partial x} & 0 & 0 & \frac{\partial N_k}{\partial x} & 0 & 0 & \frac{\partial N_l}{\partial x} & 0 & 0 \\ 0 & \frac{\partial N_i}{\partial y} & 0 & 0 & \frac{\partial N_j}{\partial y} & 0 & 0 & \frac{\partial N_k}{\partial y} & 0 & 0 & \frac{\partial N_l}{\partial y} & 0 \\ 0 & 0 & \frac{\partial N_i}{\partial z} & 0 & 0 & \frac{\partial N_j}{\partial z} & 0 & 0 & \frac{\partial N_k}{\partial z} & 0 & 0 & \frac{\partial N_l}{\partial z} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} & 0 & \frac{\partial N_j}{\partial y} & \frac{\partial N_j}{\partial x} & 0 & \frac{\partial N_k}{\partial y} & \frac{\partial N_k}{\partial x} & 0 & \frac{\partial N_l}{\partial y} & \frac{\partial N_l}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial z} & \frac{\partial N_i}{\partial y} & 0 & \frac{\partial N_j}{\partial z} & \frac{\partial N_j}{\partial y} & 0 & \frac{\partial N_k}{\partial z} & \frac{\partial N_k}{\partial y} & 0 & \frac{\partial N_l}{\partial z} & \frac{\partial N_l}{\partial y} \\ \frac{\partial N_i}{\partial z} & 0 & \frac{\partial N_i}{\partial y} & \frac{\partial N_j}{\partial z} & 0 & \frac{\partial N_j}{\partial x} & \frac{\partial N_k}{\partial z} & 0 & \frac{\partial N_k}{\partial x} & \frac{\partial N_l}{\partial z} & 0 & \frac{\partial N_l}{\partial x} \end{bmatrix}$$

Shape function of 3D elements

we can obtain in simplified form

$$\mathbf{B}^e = \begin{bmatrix} b_i & 0 & 0 & b_j & 0 & 0 & b_k & 0 & 0 & b_l & 0 & 0 \\ 0 & c_i & 0 & 0 & c_j & 0 & 0 & c_k & 0 & 0 & c_l & 0 \\ 0 & 0 & d_i & 0 & 0 & d_j & 0 & 0 & d_k & 0 & 0 & d_l \\ c_i & b_i & 0 & c_j & b_j & 0 & c_k & b_k & 0 & c_l & b_l & 0 \\ 0 & d_i & c_i & 0 & d_j & c_j & 0 & d_k & c_k & 0 & d_l & c_l \\ d_i & 0 & b_i & d_j & 0 & b_j & d_k & 0 & b_k & d_l & 0 & b_l \end{bmatrix}$$

element on tetrahedron example

Now after knowing the geometric matrix for tetrahedral element, the strain vector can be obtained from the following equation

$$\boldsymbol{\varepsilon} = \Re\mathbf{N}(x, y, z)\mathbf{u}'^{e} = \mathbf{B}^{e}\mathbf{u}'^{e}$$

which is

element on tetrahedron example

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \begin{bmatrix} b_i & 0 & 0 & b_j & 0 & 0 & b_k & 0 & 0 & b_l & 0 & 0 \\ 0 & c_i & 0 & 0 & c_j & 0 & 0 & c_k & 0 & 0 & c_l & 0 \\ 0 & 0 & d_i & 0 & 0 & d_j & 0 & 0 & d_k & 0 & 0 & d_l \\ c_i & b_i & 0 & c_j & b_j & 0 & c_k & b_k & 0 & c_l & b_l & 0 \\ 0 & d_i & c_i & 0 & d_j & c_j & 0 & d_k & c_k & 0 & d_l & c_l \\ d_i & 0 & b_i & d_j & 0 & b_j & d_k & 0 & b_k & d_l & 0 & b_l \end{bmatrix} \cdot \begin{bmatrix} u_{ix} \\ u_{iy} \\ u_{iz} \\ u_{jx} \\ u_{jy} \\ u_{jz} \\ u_{kx} \\ u_{ky} \\ u_{kz} \\ u_{lx} \\ u_{ly} \\ u_{lz} \end{bmatrix}$$

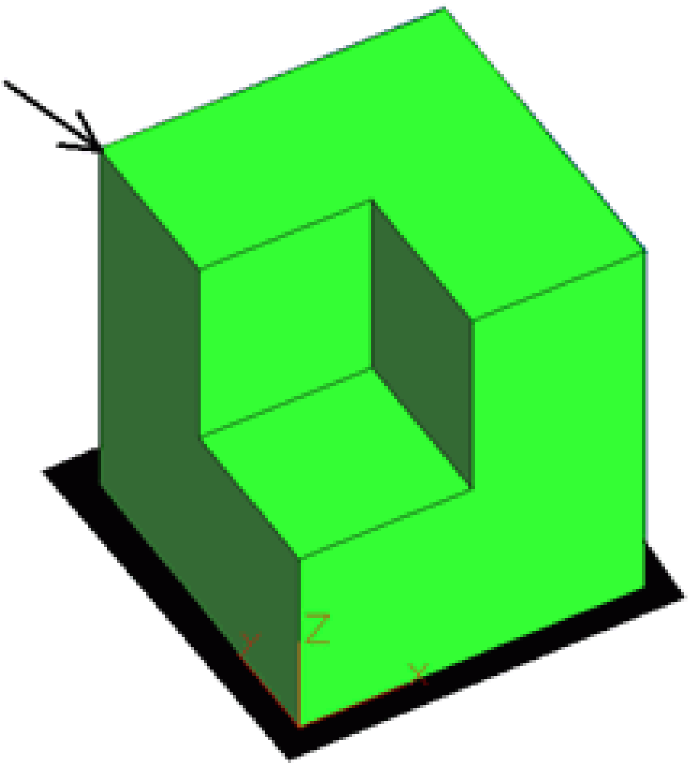
element on tetrahedron example

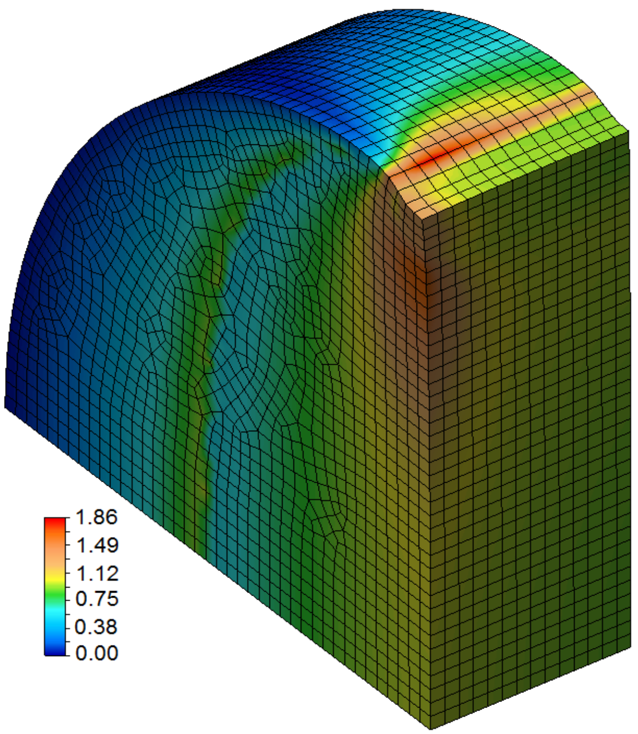
Knowing the strain vector it can be used in equation $\boldsymbol{\sigma} = \mathbf{D} \cdot \boldsymbol{\varepsilon}$ to obtain the stress vector

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-\nu} & 1 & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix}$$

formation for 3D brick models

Example of the3D brick elements use





formation for 3D brick models

There are some rules that are need to be followed:

- FEM grid must take into account the shape of the structure,
- in a hole in the model, nodes must be located so that there is no possibility to create a FEM element, but to form an empty area,
- mesh nodes must be located in location of concentrated loads,

formation for 3D brick models

mesh nodes must be located in the boundary condition points,

the edges of the grid must be located on the border between parts of elements made of different materials,

if the job is symmetrical, the mesh also should be symmetrical

formation for 3D brick models

FEM mesh should be concentrated in areas of high stress concentration and in areas of rapid change in stress (high value of gradient). Such areas are located:

- at the corners,
- around the points of application of concentrated forces,
- around the supports,

formation for 3D brick models

If the component is narrow then in cross section there should be at least four belts of elements. Only then they will be able to describe the change in stress in the cross section.