The Finite Element Method

Jerzy Podgórski

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Introduction

This book deals with the use of the finite element method (FEM is an abbreviation for the Finite Elements Method or FEA for Finite Elements Analysis) to solve linear problems of solid mechanics. We are particularly interested in static analysis of bar structures (trusses, frames), surface structures (two-dimensional plates, three-dimensional plates, shells); elements that are very often used in engineering structures. Obviously there are many books which discuss these problems, for example, the books written by the creators of FEM including Bathe (1996), Zienkiewicz (1972, 1994). In our opinion there are not enough Polish books that introduce difficult FEM problems in a simple way so that an understanding of its theoretical bases is possible for people who do not deal with structure mechanics on a daily basis. An understanding of the FEM basis is necessary for a contemporary designer who has to use sets of computer programmes in the design process and those calculated modules are just based on the finite element method. The example of a book which can be treated as a manual is the book by Rakowski and Kacprzyk (1993). This book requires some theoretical knowledge on the part of a reader. The same refers to the collective book edited by Kleiber (1995) which introduces some other computer methods used in mechanics. But the problems presented in our book are helpful to understand FEM and besides it contains the examples of exercises which can be solved by a reader without the use of a computer. A good example of a FEM handbook present on the United Kingdom market is the book written by Ross (1990).

Hence, we have decided to write a manual for engineers which is as simple as possible (but without trivialising problems) in order to simplify the study and understanding of FEM. The content of this book is based on lectures which have been given by one of the co-authors (J.P.) at the Faculty of Civil Engineering of the Technical University of Lublin since 1990. However, the content of this book has been greatly broadened and deepened in comparison with the lectures. We have also elaborated on many examples simplifying the understanding of detailed problems and algorithms of FEM.

In order to study this book, the reader should have basic knowledge of akin sciences, in particular those concerning strength of materials and the theory of elasticity. We assume that the reader is familiar with terms such as stress, strain, constitutive relations (particularly the generalised Hook’s law). References given at the end of the book elaborate on these topics in detail. Special attention should be paid to the books on the theory of elasticity written by Fung (1965), Timoshenko (1966) and the book on the strength of materials written by Dyląg et al. (1999), Gawęcki (1998), Jastrzębski et al. (1985).

The study of FEM problems requires the use of matrix algebra and we assume that the reader knows the basis of calculus. At the end of the book, in Appendix 1, one can find a short review of the most important information concerning matrix algebra necessary for reading this book.

Knowledge of numerical methods is not essential in order to understand FEM because it is linked with computer implementation of algorithms. On the other hand, this information helps when using ready-made sets applying to FEM. Since numerical methods are not always included in the programme of the university course in mathematics, we provide a review of methods of storage of stiffness matrices and of solving large sets of linear equations in Appendix. We also encourage the reader to be familiar with the book written by Georg and Liu (1981) because it is particularly devoted to these methods.

I would like to thank prof. Andrzej Garstecki and dr. Witold Kąkol from Poznań University of Technology, who have reviewed the manuscript. We would also like to thank dr. Steven Hardy from Wales University at Swansea, who has read the book to identify errors and improve the clarity of the material.

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Notation

$\mathbf{a}$, $\mathbf{b}$, $\mathbf{u}$ - column matrix – vectors

$\mathbf{A}$, $\mathbf{B}$, $\mathbf{K}$ – two-dimensional matrix

$\mathbf{u}'$, $\mathbf{K}'$, u_x – vectors, matrices and scalars in the local coordinate system of an element

$\mathbf{u}$, $\mathbf{K}$, u_X – vectors, matrices and scalars in the global coordinate system

x, y, z – axes of the local coordinate system of an element

X, Y, Z – axes of the global coordinate system

$\mathbf{q}_i$ – lower index at vectors or matrices denotes the node number i

$\mathbf{q}^e$ – upper index at vectors or matrices denotes the element number e

$u_x, u_y, u_z, \varphi_x, \varphi_y, \varphi_z$ – components of the local vector $\mathbf{u}$ in the local coordinate system

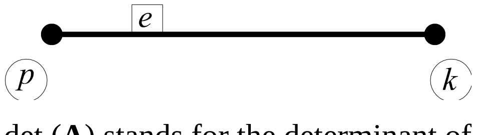
$u_X, u_Y, u_Z, \varphi_X, \varphi_Y, \varphi_Z$ – components of the global vector $\mathbf{u}$ in the global coordinate system

$\mathbf{u}_i = \begin{bmatrix} u_{iX} \\ u_{iY} \\ u_{iZ} \end{bmatrix}$ – displacement vector of node i

$\mathbf{f}_i = \begin{bmatrix} F_{iX} \\ F_{iY} \\ F_{iZ} \end{bmatrix}$ – force vector of node i

$\mathbf{u}^e = \begin{bmatrix} \mathbf{u}_p \\ \mathbf{u}_k \end{bmatrix}$ – nodal displacement vector of an element

$\mathbf{u} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}$, $\mathbf{f} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}$ – components of the vector are usually denoted by small letters just as a vector except for the nodal forces vector which is denoted by capital letters in accordance with tradition.



Element numbers are situated closer tothe first node.

det ($\mathbf{A}$) stands for the determinant of the matrix $\mathbf{A}$

$\mathbf{A}^\mathsf{T}$ – transpose of the matrix $\mathbf{A}$ which means that if $\mathbf{B} = \mathbf{A}^\mathsf{T}$, then $B_{ij} = A_{ji}$

N_N - number of nodes in a structure

N_E - number of elements in a structure

N_D - number of degrees of freedom of one node

N_{De} - number of degrees of freedom of an element

N_K - number of degrees of freedom of the whole structure

E - Young’s modulus (modulus of elasticity)

G - Kirchhoff’s modulus (modulus of elasticity in shear)

ν - Poisson’s ratio

L - length of an element

V - volume of an element

A - cross-section of a bar or the surface area of an element

J_z - inertial moment with regard to the z axis

C - torsional resistance characteristics

Introduction to the Finite Element Method

In this chapter we will discuss basic concepts and algorithms of the finite element method. We will also include necessary information regarding solid mechanics. As we have written in the Introduction, we assume that the reader knows basic issues of mechanics of materials and the theory of elasticity, therefore the information here will be only a short survey and an introduction to the matrix notation. Suitable references are given at the end of this book, in particular books written by Dylag et al. (1999), Fung (1965),Jastrzebski et al. (1985), Timoshenko and Goodier (1962).

1.1. The origin and basic concepts of the Finite Element Method

We can trace the beginnings of the finite element method to the '20s and '30s of the 20th century when authors like G.B.Maney and H.Cross in the USA and A.Ostenfeld in the Netherlands making use of findings presented in papers written by J. C. Maxwell, A. Castiliano and O. Mohr proposing a new method for solving structural mechanics problems which is now known as the displacement method.

In the middle of the 20th century J.Argyris, P.C.Pattan, S.Kelsey, M.Turner, R.Clough et al. accomplished the generalisation of this method. They did it on the basis of papers written by R.Couranta. In the '60s and '70s the finite element method was improved thanks to the publications by O.C.Zienkiewicz, Y.K.Cheung and R.L.Taylor. Thus it has become a contemporary tool used for solving issues of solid mechanics, temperature flows, fluid mechanics, electromagnetic fields and other issues.

The basic idea of the finite element method (FEM) is to search for a solution to a complex problem (which is written in the form of a differential equation) by replacing it with a simpler and similar one. It leads to the discovery of an approximate solution, the precision of which depends on the assumed approximation methods. In mechanics problems, a solution generally consists of determining displacements, strains and stresses in a continuum. These issues appear in statics and dynamics of frame structures, plates, shells and solids. The equilibrium of a body is usually written in the form of a differential equation (or a set of differential equations) which has to be realised within the body and its boundary conditions which should be realised on its surface. It is often very difficult or even impossible to find exact solutions. The finite element method proposes the following way of determining an approximate solution by Zienkiewicz (1972):

- The continuum is separated by imaginary lines or surfaces into a number of finite elements.
- The elements are assumed to be interconnected at a discrete number of nodal points situated on their boundaries. The displacements of these nodal points will be the basic unknown parameters of the problem.
- A set of functions is chosen to define uniquely the state of displacement within each finite element in terms of its nodal displacements. The displacement functions now define uniquely the state of strain within an element in terms of the nodal displacements. These strains, together with any initial strains and the constitutive properties of the material, will define the state of stress throughout the element and, hence, also on its boundaries.
- Forces concentrated at the nodes (nodal forces) which depend on nodal displacements are determined. The relationship between nodal forces and displacements is described by the element stiffness matrix.
- A set of equilibrium equations is written for all nodes, hence the problem becomes one of solving a set of algebraical equations which are often linear. Solving such a set of equations with suitable boundary conditions enables the strains and stresses within elements to be calculated.

The approximation of the solution requires solving many problems of which the selection of shape functions and discrete systems seem to be the most important ones. The person choosing a structure model (elastic, plastic, frame, plate etc.) and a discrete method should have considerable experience. In the following chapters we will present necessary information to simplify the work of less experienced users of the finite element method.

1.2. Basic assumptions and theorems of solid mechanics

Here we will present a few basic assumptions and theorems of mechanics which will be used in the subsequent chapters of this book.

1.2.1. Assumptions regarding the linear model of a structure

In this chapter and some subsequent ones we will be dealing with linear problems of mechanics. This means that the process of structural deformation can be written by linear differential equations. It involves the following consequences:

- Displacements of structure points which appear during deformation are small. Linear displacements are considerably smaller than the characteristic dimension of a structure (for example, the deflection of a beam is a few hundred times smaller than its length) and angles of rotation are considerably smaller than one (for example, a nodal angle of rotation is smaller than 0.01 rad).
- Strains are small. It enables the relationship between strains and displacements to be expressed with the help of linear equations.
- The material is linear elastic which means that it satisfies Hook's law.

It may seem that such limits which are put on both geometry of a structure and material characteristics strongly restrict the application of the model. In effect these limits are realised for many structures (they can refer to most of them), so the range of usage of the model is very wide. The reader should know this when he proceeds with the description of any real problem in terms of mechanics equations.

1.2.2. Stresses and strains

We will denote components of the stress tensor traditionally (as it occurs in most books on the finite element method). This means that components of direct stress will be denoted by letters $\sigma_x, \sigma_y, \sigma_z$ and components of shear stress by $\tau_{xy}, \tau_{xz}, \tau_{yz}$. Because of the symmetry of the stress tensor (Fung (1965), Timoshenko and Goodier (1962)), we will use only six components which when presented in a column matrix form the stress vector:

$$\sigma = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix} \tag{1}$$

Denoting the components of the strain tensor traditionally we assume the following definitions:

$$\varepsilon_x = \frac{\partial u_x}{\partial x},\ \varepsilon_y = \frac{\partial u_y}{\partial y},\ \varepsilon_z = \frac{\partial u_z}{\partial z},$$

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}, \quad \gamma_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}, \quad \gamma_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \tag{2}$$

where ε_x, ε_y, ε_z are the components of direct strain (unit elongation) and γ_{xy}, γ_{xz}, γ_{yz}, the components of shear strain (they are the angles of the non-dilatation strain), u_x, u_y, u_z are the components of the displacement vector in the Cartesian coordinate system.

We write the components of strain in the form of a column matrix - the strain vector:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} \tag{3}$$

We simplify the calculation of the internal work if we take the components of the strain vector γ_{ij} (the angles of the volumetric strain) instead of usual tensor definitions:

$$W = \int_{\mathcal{V}} \boldsymbol{\sigma}^{\mathsf{T}} \boldsymbol{\varepsilon}\, d\mathcal{V} = \int_{\mathcal{V}} \boldsymbol{\varepsilon}^{\mathsf{T}} \mathbf{D} \boldsymbol{\varepsilon}\, d\mathcal{V} \tag{4}$$

where $\mathcal{V}$ means the volume of a body.

1.2.3. Constitutive equations

As we have noted in our introductory assumptions, the relationship between the components of the stress tensor and the components of the strain tensor (that is, between $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ in our notation) is expressed by the linear equation:

$$\boldsymbol{\sigma} = \mathbf{D} \cdot \boldsymbol{\varepsilon} \tag{5}$$

$$\boldsymbol{\varepsilon} = \mathbf{D}^{-1} \cdot \boldsymbol{\sigma} \tag{6}$$

where **D** is the square matrix with dimensions 6x6 containing the material constants:

$$\mathbf{D} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \tag{7}$$

where λ and μ are the Lamé constants.

Since some other material constants like Young's modulus – E and Poisson's ratio ν are more often used, in practice we present the relationships between them and the Lamé constants by the following formulae:

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)} \tag{8}$$

The Lamé constant μ is noted by the letter G and is called Kirchhoff's (or shear) modulus.

The inverse matrix $\mathbf{D}^{-1}$ with the material constants has an unusually simple structure which is best shown by means of the constants E, ν:

$$\mathbf{D}^{-1} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} \tag{9}$$

It should be noted that matrix **D** is symmetrical which means that the dependence $\mathbf{D} = \mathbf{D}^T$ occurs. This dependence will often be used in conversions.

1.2.4. Plane stress

In two-dimensional problems of thin plates, the following simplification of the assumption is:

$$\sigma_z = 0, \quad \tau_{zx} = 0, \quad \tau_{zy} = 0, \tag{10}$$

which leads to the plane stress criterion.

If we put Eqn. (10) into Eqn. (5) taking into consideration data from Eqn. (7) we obtain:

$$\varepsilon_z = -\frac{\nu}{1-\nu}\left(\varepsilon_x + \varepsilon_y\right),\ \gamma_{zx} = 0,\ \gamma_{zy} = 0 \tag{11}$$

In plane stress, the dimensions of the stress and strain vectors and the matrix of the material constants are reduced by half and thus:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}, \quad \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} \tag{12}$$

$$\mathbf{D} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} \tag{13}$$

$$\mathbf{D}^{-1} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \tag{14}$$

1.2.5. Plane strain

In problems regarding deformations of massive buildings, the plane strain criterion is often found and it is expressed by the equations:

$$\varepsilon_z = 0, \quad \gamma_{zx} = 0, \quad \gamma_{zy} = 0 \tag{15}$$

When we insert the above equations into Eqn. (6) taking also into consideration Eqn. (9) we get the following relations:

$$\sigma_z = \nu\left(\sigma_x + \sigma_y\right), \quad \tau_{zx} = 0, \quad \tau_{zy} = 0 \tag{16}$$

This is called the plane strain.

After taking into consideration the above Eqn. (15) and (16), we can notice that the relationship between the reduced stress and strain vectors Eqn. (12) leads to the following matrix of elastic constants:

$$\mathbf{D} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix} \tag{17}$$

$$\mathbf{D}^{-1} = \frac{1-\nu^2}{E}\begin{bmatrix} 1 & \dfrac{-\nu}{1-\nu} & 0 \\ \dfrac{-\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \dfrac{2}{1-\nu} \end{bmatrix} \tag{18}$$

1.2.6. Equilibrium equations

The condition of equilibrium for a fixed body is satisfied when the following six equations called equilibrium equations take place:

$$\sum_{i=1}^{n} \mathbf{P}_i = \mathbf{0}, \ \sum_{i=1}^{n} \mathbf{M}_i = \mathbf{0} \tag{19}$$

which can be written as:

$$\begin{gathered} \sum_{i=1}^{n} P_{Xi} = 0; \quad \sum_{i=1}^{n} P_{Yi} = 0; \quad \sum_{i=1}^{n} P_{Zi} = 0; \\ \sum_{i=1}^{n} M_{Xi} = 0; \quad \sum_{i=1}^{n} M_{Yi} = 0; \quad \sum_{i=1}^{n} M_{Zi} = 0 \end{gathered} \tag{20}$$

where P_{Xi}, P_{Yi}, P_{Zi} are the components of the force $\mathbf{P}_i$ and M_{Xi}, M_{Yi}, M_{Zi} are the moments of this force in relation to the axes of a coordinate system and n is the number of forces.

When a set of forces is contained in, for example, the plane XY, then equilibrium Eqn. (20) are reduced to the following three equations:

$$\sum_{i=1}^{n} P_{Xi} = 0; \quad \sum_{i=1}^{n} P_{Yi} = 0; \quad \sum_{i=1}^{n} M_{Zi} = 0 \tag{21}$$

1.2.7. The principle of virtual work

Equilibrium Eqn. (19) define conditions for a set of forces acting on a rigid body. In the case of an elastic body which deforms due to forces acting on it we have to determine conditions for external forces, as well. This can be done by using the principle of virtual work which says that the external work due to virtual displacements is equivalent to the increase of the potential energy of the internal forces:

$$\sum_{i=1}^{n} \mathbf{P}_i \cdot \bar{\mathbf{u}}_i = E_\sigma \tag{22}$$

where $\overline{\mathbf{u}}_i$ is the vector of the virtual displacement at the point i, the dot means the scalar product of the vector of the force $\mathbf{P}_i$ and the vector of the virtual displacement $\overline{\mathbf{u}}_i$, E_σ - is the potential energy of internal forces:

$$E_\sigma = \int_{\mathcal{V}} \boldsymbol{\sigma}^{\mathrm{T}} \overline{\boldsymbol{\varepsilon}} \, d\mathcal{V} = \int_{\mathcal{V}} \overline{\boldsymbol{\varepsilon}}^{\mathrm{T}} \boldsymbol{\sigma} \, d\mathcal{V} \tag{23}$$

In Eqn. (23) $\overline{\boldsymbol{\varepsilon}}$ denotes the strain vector which results from the virtual displacement $\overline{\mathbf{u}}$.

The virtual displacement must satisfy the following conditions from Nowacki (1976):

- it should be independent of forces acting on a solid,
- it should be consistent with the constraints so that it is kinematically allowable,
- it should be independent of time.

Eqn. (22) will be used many times in different forms in the subsequent chapters of this book.

1.2.8. Clapeyron's theorem

Changing virtual displacements into the real ones in Eqn. (22) and (23) we obtain:

$$\sum_{i=1}^{n} \mathbf{P}_i \cdot \mathbf{u}_i = \int_{\mathcal{V}} \boldsymbol{\sigma}^{\mathrm{T}} \boldsymbol{\varepsilon} \, d\mathcal{V} = \int_{\mathcal{V}} \boldsymbol{\varepsilon}^{\mathrm{T}} \boldsymbol{\sigma} \, d\mathcal{V} \tag{24}$$

The above equation expresses the content of Clapeyron's theorem which says that for the elastic body in equilibrium the work of external forces is equal to the potential energy of internal forces (elastic energy). Moreover, the elastic body has to satisfy the conditions described by Gawęcki (1998) and Jastrzębski et al. (1985):

- material of which the body is composed reacts according to Hook's law,
- body does not possess the boundary conditions which depend on the deformation of a structure,
- body temperature is constant
- there are no initial stresses and strains.

Bodies which satisfy these conditions are called Clapeyron's bodies.

1.2.9. The Betti reciprocal theorem of work and the Maxwell reciprocal theorem of displacements.

Let us insert the constitutive relation into Eqn. (22) expressing the principle of virtual work Eqn. (5). Thus we obtain:

$$\sum_{i=1}^{n} \mathbf{P}_i \cdot \mathbf{u}_i = \int_{\mathcal{V}} \boldsymbol{\sigma}^{\mathsf{T}} \boldsymbol{\varepsilon} \, d\mathcal{V} = \int_{\mathcal{V}} (\mathbf{D}\boldsymbol{\varepsilon})^{\mathsf{T}} \boldsymbol{\varepsilon} \, d\mathcal{V} = \int_{\mathcal{V}} \boldsymbol{\varepsilon}^{\mathsf{T}} \mathbf{D} \boldsymbol{\varepsilon} \, d\mathcal{V} \tag{25}$$

In the above equations we have made use of the symmetry of the matrix of elastic constants $\mathbf{D}=\mathbf{D}^{\mathsf{T}}$.

Below we will apply the principle of virtual work in a different way, namely we attach virtual loads (a set of forces $\overline{\mathbf{P}}_j$) acting at the same nodes as the actual loads, but of a different value and direction. The work done by these forces for the actual displacement is equal to:

$$\sum_{j=1}^{n} \mathbf{P}_j \cdot \overline{\mathbf{u}}_j = \int_{\mathcal{V}} \boldsymbol{\sigma}^{\mathsf{T}} \boldsymbol{\varepsilon} \, d\mathcal{V} = \int_{\mathcal{V}} (\mathbf{D}\boldsymbol{\varepsilon})^{\mathsf{T}} \boldsymbol{\varepsilon} \, d\mathcal{V} = \int_{\mathcal{V}} \boldsymbol{\varepsilon}^{\mathsf{T}} \mathbf{D} \boldsymbol{\varepsilon} \, d\mathcal{V} \tag{26}$$

The right hand sides of Eqn.(25) and (26) are identical which can be simply checked by direct calculations. Hence, we obtain the equation:

$$\sum_{i=1}^{n} \mathbf{P}_i \cdot \overline{\mathbf{u}}_i = \sum_{i=1}^{n} \overline{\mathbf{P}}_i \cdot \mathbf{u}_i \tag{27}$$

which expresses the reciprocal theorem of work formulated by E.Betti in the nineteenth century.

This theorem can be written as follows (Nowacki (1976)):

The set of forces $\mathbf{P}_i$ does the same work at the displacements induced by the set of forces $\mathbf{P}_j$, as the set of forces $\mathbf{P}_j$ does at displacements induced by forces $\mathbf{P}_i$.

If we bring down both sets of forces to single unit forces acting at the point *a*, we obtain:

$$\mathbf{1}_a \cdot \overline{\mathbf{u}}_a = \overline{\mathbf{1}}_a \cdot \mathbf{u}_a \tag{28}$$

This relationship is called the reciprocal theorem of displacements and was formulated by J.C.Maxwell in 1864.

1.3. Algorithm of the Finite Element Method

The finite element method as a computer method is characterised by a strictly defined and simple algorithm. We will show the most important stages of this algorithm. Some of them will be discussed in detail in further parts of this book.

A. *Discretization*

At this stage, the division of a structure into finite elements is done. In the case of frameworks it is often obvious since every straight segment of a bar becomes an element. In the case of 2D surfaces, we divide their area into triangular and/or quadrangular elements and in the case of solids we divide them into tetrahedral and hexahedronal (brick) elements.

At this stage, we decide about points of elements contacts, give coordinates of the nodes and state the manner of connection between nodes and elements.

B. *Calculation of element stiffness matrices*

On the basis of material properties and topological data given in the first stage matrices expressing relationships between nodal forces and nodal displacements of an element are formed.

C. *Aggregation (construction) of a global stiffness matrix*

Now element stiffness matrices are divided into blocks which merge into a global stiffness matrix for which the information about construction topology is used. Modifications taking into consideration boundary conditions are often introduced into the global matrix at that stage.

D. *Construction of a global loads vector*

Here we calculate load vectors of elements which, after being divided into blocks, are inserted into the global vector of nodal loads. When the global vector is built, then its components should be modified with regard to boundary conditions.

E. *Solution of a set of equations*

At this stage, a set of linear equations will be solved. In effect, we will obtain the nodal displacements of a structure.

F. *Calculation of internal forces and reactions*

If we obtain displacements, we can then calculate strains, stresses and internal forces in a structure. After having calculated element nodal forces, reactions at constraints (supports) of the construction can also be calculated.

The systems of FEM usually have a modular structure. Individual stages of the algorithm are solved by specialised modules of the system.

The first stage (A) complemented by defining material properties and describing construction loads is called a preprocessor. In old systems, this stage depended on manual creation of a data file (input file). At present, such a situation occurs very rarely because manually inputting data for the typical problem of FEM (covering a few thousands of nodes) is very hard work. Contemporary preprocessors are usually graphic programmes equipped with tools simplifying the generation of element meshes.

Stages (B), (C), (D), and (E) are usually performed by the module called a processor. Apart from the operations mentioned above, the processor often deals with a suitable arrangement of equations in order to reduce the amount of memory for the storage of the stiffness matrix and to accelerate the process of solving systems of equations.

The sixth stage (F) complemented by graphical output is undertaken by a postprocessor. A large amount of results that we get after solving any system of equations and calculating internal forces is very difficult to interpret without using graphical techniques. Contemporary FEM systems are equipped with a graphic postprocessor producing colour maps of stresses, displacements and other parameters which simplify analyses.

Although visual techniques are strongly linked with the finite element method, they are not a part of this course; hence they are not described in this book. We will concentrate on the processor and parts of the postprocessor.

1.3.1. Creation of element stiffness matrices

As we have already noted in this chapter (point 1.1), we assume that after having divided the structure into finite elements, these elements are only in contact with each other at nodes. It will be convenient if we imagine a node as a material particle moving during the deformation process caused by external loads affecting the structure (forces, temperatures, etc.). We can describe the movement of a node by giving the components of the displacement vectors. We will be interested in different types of motion according to the element type. In some cases they will be displacements (in truss elements, two-dimensional plane elements, solids), in other cases there are rotations (in beams, frames, plates, shells). All necessary components of a nodal displacement create the system of parameters called degrees of freedom. We will mark the number of degrees of freedom as N_D.

In Table 1 there is information on the number of degrees of freedom for nodes of typical engineering structures. Degrees of freedom are given as components of displacement vectors in the Cartesian coordinate system.

Table 1. The degree types of freedom for elements.

Type of structure	Number of degrees of freedom	Displacements			Rotations		
	N_D	u_x	u_y	u_z	φ_x	φ_y	φ_z
plane truss	2	•	•				
space truss	3	•	•	•			
plane frame	3	•	•				•
space frame	6	•	•	•	•	•	•
grillwork	3			•	•	•	
two-dimensional	2	•	•				
plate	3			•	•	•	
shell	6	•	•	•	•	•	•
solid (brick)	3	•	•	•			

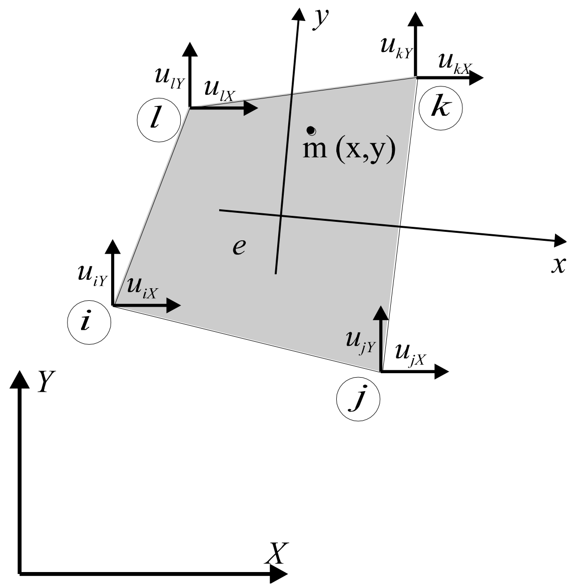


Figure 1. The plane finite element with four nodes.

Let us imagine some quadrilateral element (for convenience we will take a plane element which is easy to draw) having number e (Figure 1). The nodes of this element are locally numbered: i, j, k, l and they have their global numbers respectively: n_i, n_j, n_k, n_l. Nodal coordinates are always given in the global coordinate system XY , but for convenience we use any local coordinate system while forming an element stiffness matrix. The local coordinate system is chosen at random.

We group nodal displacements in the displacement vector:

$$\mathbf{u}_i = \begin{bmatrix} u_{iX} \\ u_{iY} \end{bmatrix}, \quad \mathbf{u}_j = \begin{bmatrix} u_{jX} \\ u_{jY} \end{bmatrix}, \quad \mathbf{u}_k = \begin{bmatrix} u_{kX} \\ u_{kY} \end{bmatrix}, \quad \mathbf{u}_l = \begin{bmatrix} u_{lX} \\ u_{lY} \end{bmatrix} \tag{29}$$

The set of all nodal displacements of an element forms the vector of nodal displacements of an element:

$$\mathbf{u}^e = \begin{bmatrix} \mathbf{u}_i \\ \mathbf{u}_j \\ \mathbf{u}_k \\ \mathbf{u}_l \end{bmatrix} = \begin{bmatrix} u_{iX} \\ u_{iY} \\ u_{jX} \\ u_{jY} \\ u_{kX} \\ u_{kY} \\ u_{lX} \\ u_{lY} \end{bmatrix} \tag{30}$$

The displacement of a certain point m within the element is written in the form of the vector:

$$\mathbf{u}(X,Y) = \begin{bmatrix} u_X(X,Y) \\ u_Y(X,Y) \end{bmatrix} \tag{31}$$

If the components of vectors are defined in a local coordinate system, then we will denote them as the sign ' (prim), for instance:

$$\mathbf{u}'(x,y) = \begin{bmatrix} u_x(x,y) \\ u_y(x,y) \end{bmatrix} \tag{31a}$$

Similar notation can be used in Eqn. (29) and (30) but for the time being we will use only global relationships for convenience.

Now we assume that the displacement of some point m depends on nodal displacements of an element:

$$\mathbf{u}(x,y) = \mathbf{N}^e(x,y)\mathbf{u}^e \tag{32}$$

where $\mathbf{N}(x,y)$ is the matrix component which depends on the coordinates of a point. The dimensions of the matrix $\mathbf{N}(x,y)$ depend on element type. The number of rows of the matrix $\mathbf{N}(x,y)$ is equal to the number of degrees of freedom of the point m and the number of columns, represents the number of degrees of freedom of the element. In our example where the point has two degrees of freedom and the element has 4x2=8 degrees of freedom, the matrix $\mathbf{N}(x,y)$ has two rows and eight columns.

Thus, it will be convenient to present Eqn. (32) in a developed form:

$$\mathbf{u}(x,y) = \begin{bmatrix} \mathbf{N}_i(x,y) & \mathbf{N}_j(x,y) & \mathbf{N}_k(x,y) & \mathbf{N}_l(x,y) \end{bmatrix} \begin{bmatrix} \mathbf{u}_i \\ \mathbf{u}_j \\ \mathbf{u}_k \\ \mathbf{u}_l \end{bmatrix} \tag{32a}$$

where matrices $\mathbf{N}_i(x,y)$ ... $\mathbf{N}_l(x,y)$ are quadratic matrices containing functions which show the influence of the displacements of nodes i ... l on the displacement of the point m. In the finite element method, these functions are known as *shape functions* or *displacement functions* and they are very important for the formulation of FEM equations. Matrices $\mathbf{N}_i(x,y)$ ... $\mathbf{N}_l(x,y)$ are called matrices of shape functions of nodes i ... l and the matrix $\mathbf{N}^e(x,y)$ is the matrix of shape functions of an element.

It is obvious that shape functions should fulfil some conditions to be useful for the approximation of the field of an element displacement. If we imagine that the point m is at a node, then its displacements should be equal to the displacements of this node, but the displacements of other nodes should not have any influence on them (Figure 2).

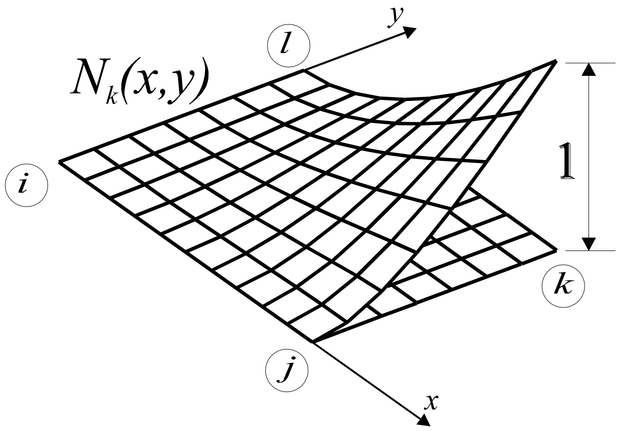


Figure 2. The deformation of the element surface whose the k node is displaced by a unit in the direction perpendicular to this element.

This condition can be expressed in the following way:

$$N_p\left(x_q, y_q\right) = \delta_{pq} \tag{33}$$

where δ_{pq} is Kronnecker's delta: $\delta_{pq} = \begin{cases} 1 & \text{- when } p{=}q, \\ 0 & \text{- when } p \neq q \end{cases}$

and p and q represent any local number of nodes i ... l.

Conditions of type Eqn. (33) allow us to determine the coefficients of shape functions. We will consider some other conditions which have to be fulfilled by functions $N_p(x,y)$ in later parts of this chapter.

Substituting Eqn. (32) for Eqn. (5) we calculate the components of the element strain vector:

$$\boldsymbol{\varepsilon} = \mathbf{D} \cdot \mathbf{N}^e\left(x, y\right)\mathbf{u}^e \tag{34}$$

where **D** is the matrix with dimensions $3 \times N_D$ for both plane stress and plane strain or $6 \times N_D$ for three-dimensional problems (N_D is the number of degrees of freedom of a node) containing differential operators coming from the definition of strain Eqn. (2)

For a two-dimensional problem, N_D=2 and the matrix of differential operators has the following form:

$$\mathscr{D} = \begin{bmatrix} \partial_x & 0 \\ 0 & \partial_y \\ \partial_y & \partial_x \end{bmatrix}, \tag{35}$$

where symbol ∂_x signifies differentiation with respect to x: $\partial_x = \dfrac{\partial}{\partial x}$ and ∂_y with respect to y.

We assume the notations:

$$\mathscr{D} \cdot \mathbf{N}^e\left(x, y\right) = \mathbf{B}^e\left(x, y\right) \tag{36}$$

and consistently

$$\begin{aligned} &\mathscr{D} \cdot \mathbf{N}_i\left(x, y\right) = \mathbf{B}_i\left(x, y\right), \\ &\vdots \\ &\mathscr{D} \cdot \mathbf{N}_l\left(x, y\right) = \mathbf{B}_l\left(x, y\right). \end{aligned} \tag{37}$$

They simplify further transformations.

After taking into consideration these notations, relation Eqn. (34) can be presented as:

$$\boldsymbol{\varepsilon} = \mathbf{B}^e(x,y)\mathbf{u}^e, \tag{38}$$

The matrix $\mathbf{B}(x,y)$ has dimensions $3\times n_D^e$ (or $6\times n_D^e$ for three-dimensional problems of stress).

For a quadrilateral element in a two-dimensional problem, matrix $\mathbf{B}(x,y)$ has dimensions 3×8. As with matrix $\mathbf{N}(x,y)$, we now similarly divide the matrix $\mathbf{B}(x,y)$ into blocks:

$$\mathbf{B}^e(x,y) = \left[\mathbf{B}_i(x,y) \quad \mathbf{B}_j(x,y) \quad \mathbf{B}_k(x,y) \quad \mathbf{B}_l(x,y)\right] \tag{39}$$

Matrices $\mathbf{B}_i$ ... $\mathbf{B}_l$ are matrices containing strain shape functions of nodes i ... l, and $\mathbf{B}^e(x,y)$ is the matrix containing strain shape functions of the element e.

Here we replace reactions between nodes and elements by concentrated forces. The scheme of these reactions is shown in Figure 3.

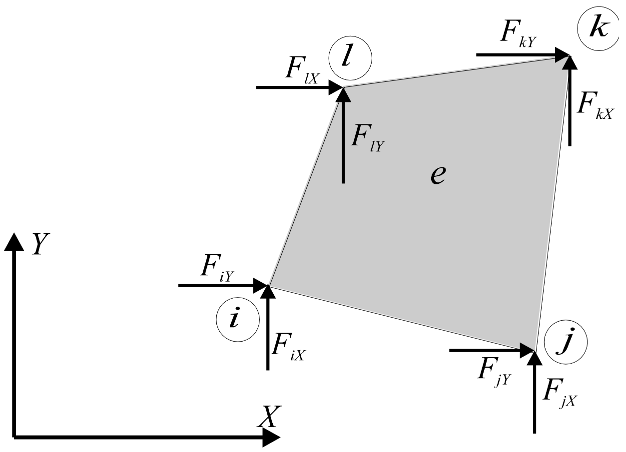


Figure 3.The arrangement of nodal forces over the element.

Now we collect the components of nodal forces into the nodal force vector:

$$\mathbf{f}_i = \begin{bmatrix} F_{iX} \\ F_{iY} \end{bmatrix},\ \mathbf{f}_j = \begin{bmatrix} F_{jX} \\ F_{jY} \end{bmatrix},\ \mathbf{f}_k = \begin{bmatrix} F_{kX} \\ F_{kY} \end{bmatrix},\ \mathbf{f}_l = \begin{bmatrix} F_{lX} \\ F_{lY} \end{bmatrix} \tag{40}$$

and the forces acting on an element into the nodal force vector of an element:

$$\mathbf{f}^e = \begin{bmatrix} \mathbf{f}_i \\ \mathbf{f}_j \\ \mathbf{f}_k \\ \mathbf{f}_l \end{bmatrix} = \begin{bmatrix} F_{iX} \\ F_{iY} \\ F_{jX} \\ F_{jY} \\ F_{kX} \\ F_{kY} \\ F_{lX} \\ F_{lY} \end{bmatrix} \tag{41}$$

Let us look for the relationship between nodal forces $\mathbf{f}^e$ and nodal displacements $\mathbf{u}^e$.

We apply the principle of virtual work Eqn. (22) treating the nodal forces as the external loads on an element. The element is loaded both on its inside and boundary and we denote the load which depends on the coordinates of a point as follows:

$$\mathbf{q}(x,y) = \begin{bmatrix} q_x(x,y) \\ q_y(x,y) \end{bmatrix} \tag{42}$$

We divide constitutive Eqn. (5) (or for instance Eqn. (12) and (13) for plane stress) into parts in order to consider initial strains and stresses:

$$\boldsymbol{\sigma} = \mathbf{D}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_\mathbf{o}) + \boldsymbol{\sigma}_\mathbf{o} \tag{43}$$

where $\boldsymbol{\varepsilon}_o$ is the initial strain vector (for example, caused by temperature loads) and $\boldsymbol{\sigma}_\mathbf{o}$ is the initial stress vector (eg. residual stresses).

Now we re-write Eqn. (22) expressing the equality of external and internal work for the element in equilibrium:

$$\left(\mathbf{u}^e\right)^{\mathsf{T}} \mathbf{f}^e + \int_{\mathcal{A}} \mathbf{u}(x,y)^{\mathsf{T}} \mathbf{q}(x,y) d\mathcal{A} = \int_{\mathcal{V}} \boldsymbol{\varepsilon}^{\mathsf{T}} \boldsymbol{\sigma}\, d\mathcal{V} \tag{44}$$

The left hand side of this equation represents external work while the right hand side denotes internal work for this element, **A** represents the surface of an element and **V**is its volume. We use Eqn. (32), (38) and (43) in the above equation:

$$\left(\mathbf{u}^e\right)^{\mathsf{T}} \mathbf{f}^e + \int_{\mathcal{A}} \left(\mathbf{N}^e \mathbf{u}^e\right)^{\mathsf{T}} \mathbf{q} d\mathcal{A} = \int_{\mathcal{V}} \left(\mathbf{B}^e \mathbf{u}^e\right)^{\mathsf{T}} \left[\mathbf{D}\left(\mathbf{B}^e \mathbf{u}^e - \boldsymbol{\varepsilon}_\mathbf{o}\right) + \boldsymbol{\sigma}_\mathbf{o}\right] d\mathcal{V} \tag{45}$$

After the transformation we obtain its final form as follows:

$$\mathbf{f}^e = \mathbf{K}^e \mathbf{u}^e - \mathbf{f}_q^e - \mathbf{f}_{\varepsilon_o}^e + \mathbf{f}_{\sigma_o}^e \tag{46}$$

where the following values have been noted:

- nodal forces vector due to external loads:

$$\mathbf{f}_q^e = \int_{\mathcal{A}} \left(\mathbf{N}^e\right)^{\mathsf{T}} \mathbf{q}\, d\mathcal{A} \tag{47}$$

- nodal forces vector due to initial strain:

$$\mathbf{f}_{\varepsilon_o}^e = \int_{\mathcal{V}} \left(\mathbf{B}^e\right)^{\mathsf{T}} \mathbf{D}\boldsymbol{\varepsilon}_{\mathbf{o}} d\mathcal{V} \tag{48}$$

- odal forces vector due to initial stress:

$$\mathbf{f}_{\sigma_o}^e = \int_{\mathcal{V}} \left(\mathbf{B}^e\right)^{\mathsf{T}} \boldsymbol{\sigma}_{\mathbf{o}} d\mathcal{V} \tag{49}$$

- element stiffness matrix:

$$\mathbf{K}^e = \int_{\mathcal{V}} \left(\mathbf{B}^e\right)^{\mathsf{T}} \mathbf{D}\mathbf{B}^e d\mathcal{V} \tag{50}$$

Thus calculated nodal force vectors contain forces acting on the element. They should be marked with the negative sign when forming equilibrium equations.

The matrix $\mathbf{K}^e$ can be divided into a block of quadratic matrices $\mathbf{K}_{pq}^e$ describing the influence of the displacement of the node *q* on the forces at the node *p*:

$$\mathbf{K}_{pq}^e = \int_{\mathcal{V}} \left(\mathbf{B}_p^e\right)^{\mathsf{T}} \mathbf{D}\mathbf{B}_q^e d\mathcal{V} \tag{51}$$

There are 4×4=16 blocks in the stiffness matrix of the element with four nodes (Figure 3). Since the stiffness matrix is symmetrical, it means that $\mathbf{K}^e = \left(\mathbf{K}^e\right)^{\mathsf{T}}$ which comes from Eqn. (50) and it is a simple consequence of the Betti reciprocal theorem of work; then blocks $\mathbf{K}_{pq}^e$ have to realise the conditions:

$$\mathbf{K}_{qp}^e = \left(\mathbf{K}_{pq}^e\right)^{\mathsf{T}} \tag{52}$$

Eqn. (50) or (51) represents a key step in formulating equilibrium equations of the structure but the stiffness matrix has not always been determined this way. For simple elements such as a truss element or a frame element, some other ways (sometimes simpler) of obtaining relation Eqn. (46) exist. We will show these in next chapters.

If all transformations leading to Eqn. (50) have been done in the local coordinate system (*xyz*), then the resulting stiffness matrix should be transformed to the global coordinate system (*XYZ*). This transformation is achieved by multiplying the matrix

$\mathbf{K}^{e\,\prime}$ (sign prim denotes a matrix in the local coordinate system) by the transformation matrix of the element. The detailed structure of these matrices is elaborated on in Chapters 2, 3 and 4, here we simply illustrate the transformation:

$$\mathbf{K}^{e} = \mathbf{R}^{e}\mathbf{K}'^{e}\left(\mathbf{R}^{e}\right)^{\mathrm{T}} \tag{53}$$

$$\text{where } \mathbf{R}^{e} = \begin{bmatrix} \mathbf{R}_i & & & \\ & \mathbf{R}_j & & \\ & & \mathbf{R}_k & \\ & & & \ddots \end{bmatrix} \tag{54}$$

$\mathbf{R}_i$ ... $\mathbf{R}_k$ - transformation matrices of nodes i ... k. The transformation matrices of the nodes contain cosines of angles between the axes of the global and local coordinate systems:

$$\mathbf{R}_i = \begin{bmatrix} C_{xX} & C_{xY} & C_{xZ} \\ C_{yX} & C_{yY} & C_{yZ} \\ C_{zX} & C_{zY} & C_{zZ} \end{bmatrix} \tag{55}$$

where, for instance, $C_{xY} = \cos(\alpha_{xY})$, etc., α_{xY} is the angle between the x axis of the local coordinate system and the Y axis of the global system.

1.3.2. Aggregation (construction) of a global stiffness matrix

Relation Eqn. (46) allows us to write equilibrium equations of a node in the form containing nodal displacements as unknown.

Let us imagine a node as an independent part of a construction and disconnect elements from nodes in order to show nodal forces (Figure 4).

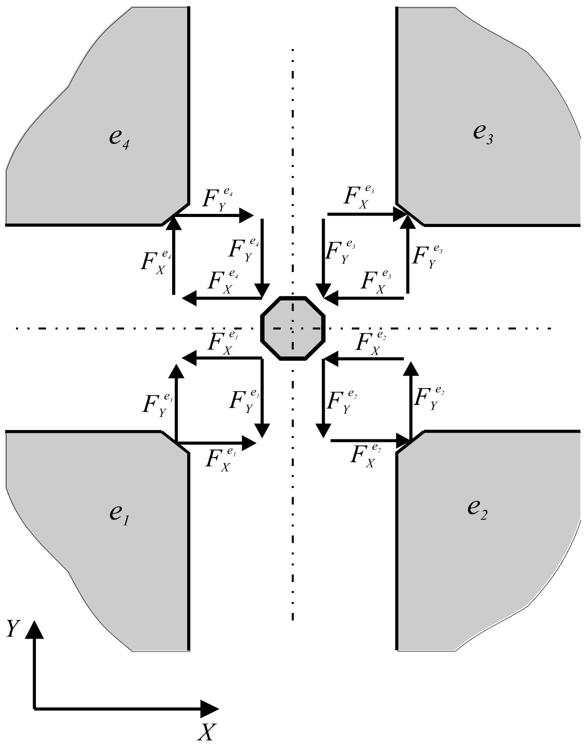


Figure 4. The senses of forces representing the interaction between elements and nodes.

We write a set of equilibrium equations of the node in the scalar form:

$$\sum_{k=1}^{E_n} F_X^{e_k} = 0 \ , \ \sum_{k=1}^{E_n} F_Y^{e_k} = 0 \ , \ \sum_{k=1}^{E_n} F_Z^{e_k} = 0 \tag{56a}$$

For the nodes with rotational degrees of freedom, the equilibrium equations of moments will be necessary:

$$\sum_{k=1}^{E_n} M_X^{e_k} = 0 \ , \ \sum_{k=1}^{E_n} M_Y^{e_k} = 0 \ , \ \sum_{k=1}^{E_n} M_Z^{e_k} = 0 \tag{56b}$$

In Eqn. (56) summation is required for all elements connected to the node, hence indices e_1, e_2 ... e_{E_n} are numbers of elements connected to the node, E_n is the number of elements connected to the node n. We insert relationship Eqn. (46) into Eqn. (56) remembering to change the sign of the nodal forces coming from the change of sense of the forces acting on the element and node (Figure 4):

$$-\sum_{k=1}^{E_n} \mathbf{f}_n^{e_k} = 0 \tag{57}$$

In this equation, symbol $\mathbf{f}_n^{e_k}$ defines only these components of vector $\mathbf{f}^{e_k}$ which act on the node *n*. We convert this equation into a more convenient form:

$$\sum_{k=1}^{E_n} \mathbf{K}_n^{e_k} \mathbf{u}_n^{e_k} = \mathbf{p}_n^{e_k} \tag{58}$$

where $\mathbf{p}^e = \mathbf{f}_q^e + \mathbf{f}_{\varepsilon_o}^e - \mathbf{f}_{\sigma_o}^e$ is the vector of the nodal forces due to external loads, initial strains and stresses.

Arranging equations for every node of the structure similar to Eqn. (56), we obtain a set of equations which allow us to calculate nodal displacements for this structure. Since summation is done for the elements in Eqn. (56) (the force vectors which belong to this node), formation of a set of equations based on the equilibrium of successive nodes is not effective.

Ordering nodes and degrees of freedom is necessary for this operation. So far we have used local numbers for nodes of elements *i*, *j*, *k*, *l* .., but introducing global numeration of nodes is necessary while building the global set of equations. Let n_i stand for a global number of the node represented by the local number *i* and let s_p be a global number of degrees of freedom represented by the local number *p*. Now we form a rectangular matrix of connections of the element e – $\mathbf{A}^e$. The number of rows of the matrix $\mathbf{A}^e$ is equal to the global number of degrees of freedom of the structure N_k , the number of columns is equal to the number of degrees of freedom of the element e – N_D^e. Most components of the matrix $\mathbf{A}^e$ are equal to zero apart from the components having the value of 1 which are situated in rows s_p and columns *p*. Hence, the structure of the matrix $\mathbf{A}^e$ contains information about connections between the element and nodes or being more exact about the relationship between the degree of freedom of the element and the global degree of freedom of the structure. The formation of the connection matrix can be most easily studied on the following example.

Figure 5presents a plate divided into five triangular elements. The plate has six nodes numbered from 1 to 6, every element has a local notation of nodes *i*, *j*, *k*. Table 2 shows global numeration of degrees of freedom of a two-dimensional element of the plate.

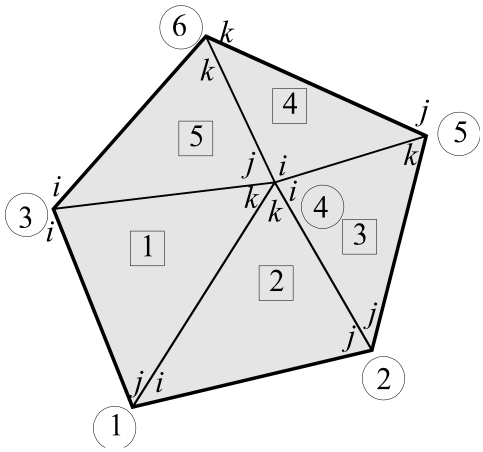


Figure 5. The exemplary discretisation of the 2D membrane into five finite elements.

Table 2. The global numbers of degrees of freedom for nodes in the plate (Figure 5).

Node number	Global numbers of degrees of freedom of nodes	
n	u_{nX}	u_{nY}
1	1	2
2	3	4
3	5	6
4	7	8
5	9	10
6	11	12

Table 3.The global numbers of degrees of freedom for elements in the plate (Figure 5).

Element number	Global numbers of degrees of freedom of element s_p – allocation vector					
e	u_{iX}	u_{iY}	u_{jX}	u_{jY}	u_{kX}	u_{kY}
	1	2	3	4	5	6
1	5	6	1	2	7	8
2	1	2	3	4	7	8
3	7	8	3	4	9	10
4	7	8	9	10	11	12
5	5	6	7	8	11	12

Table 3shows the dependence between local and global degrees of freedom. Hence the connection matrix created for element No 3 will have the following form:

$$\mathbf{A}^3 = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cccccc} \; & \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; & \; \\ & & 1 & & & \\ & & & 1 & & \\ \; & \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; & \; \\ 1 & & & & & \\ & 1 & & & & \\ & & & & 1 & \\ & & & & & 1 \\ \; & \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; & \; \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{array}$$

where all zero elements are neglected for clarity.

Multiplying the nodal force vector of an element by the connection matrix causes the transfer of suitable blocks of the local vector to the global vector. Now simple addition of these vectors is possible:

$$\sum_{e=1}^{N_E} \mathbf{A}^e \mathbf{f}^e = \sum_{e=1}^{N_E} \mathbf{A}^e \mathbf{K}^e \mathbf{u}^e = \sum_{e=1}^{N_E} \mathbf{A}^e \mathbf{p}^e \tag{59}$$

Here it is necessary to express the nodal displacement vector of elements by means of the global vector:

$$\mathbf{u}^e = \left(\mathbf{A}^e\right)^{\mathrm{T}} \mathbf{u} \,,$$

which should be put into Eqn. (59). Finally, we obtain the system of equations in the form:

$$\sum_{e=1}^{N_E} \mathbf{A}^e \mathbf{K}^e \left(\mathbf{A}^e\right)^{\mathrm{T}} \mathbf{u} = \sum_{e=1}^{N_E} \mathbf{A}^e \mathbf{p}^e \tag{60}$$

or in a shorter form

$$\mathbf{Ku} = \mathbf{p} \, . \tag{61}$$

Matrix $\mathbf{K} = \sum_{e=1}^{N_E} \mathbf{A}^e \mathbf{K}^e \left(\mathbf{A}^e\right)^{\mathrm{T}}$ is called the global stiffness matrix of a structure, vector $\mathbf{p} = \sum_{e=1}^{N_E} \mathbf{A}^e \mathbf{p}^e$ is the global vector of nodal forces of the structure, the vector $\mathbf{u}$ containing the displacement of all nodes is the global displacement vector.

A similar method of aggregation is described in the book written by Rakowski and Kacprzyk (1993) where matrix $\mathbf{A}^{\mathrm{T}}$ is called the connection matrix.

The method of aggregation using the connection matrix is not suitable for computer implementation because it uses the big matrix $\mathbf{A}^e$. It is more effective to exploit information which is contained in allocation vectors. Vectors for the previous example are included in Table 3. The aggregation method using allocation vectors will be presented in the second chapter in sections devoted to building the stiffness matrix of a truss.

1.3.3. Remarks regarding the shape functions of an element

Functions approximating the displacement field within elements which are in fact shape functions described in Sec.1.3.1 cannot be chosen in freely. They should fulfil some conditions which decide about the quality of these functions or their usefulness for approximation of displacements, strains and stresses. We quote these criteria after Zienkiewicz (1972).

A. *Criteria of rigid body movements*

The displacement function chosen should be in such a way that it should not permit straining of an element to occur when the nodal displacements are caused by a rigid body displacement.

B. *Criterion of strain stability*

The shape function should enable the constant field of strains in an element to appear.

C. *Criterion of strain agreement*

The displacement functions should be so chosen that the strains at the interface between elements are finite.

Criteria (A) and (B) seem to be obvious. Since some components of strain (or stress) can be zero, then approximation functions should be able to reproduce these problems. Constant and linear parts of polynomials which we often use to build a shape structure, assure realisation of conditions (A) and (B). Criterion (B) is the generalisation of criterion (A) and it was formulated by Bazeley, Cheung, Irons and Zienkiewicz (1972, 1994) in 1965.

Criterion (C) requires that shape functions should assure continuity of derivatives to the degree which is lower by one than differential operators being in the matrix $\mathscr{D}$ (comp. Eqn. (34)). We explain this using the following example. In the two-dimensional problem of a plate, the strains are defined by the first derivation of the displacement function (comp. Eqn. (34) and (35)), because the displacement field has to be continuous on the boundary between elements and displacements functions have to be of class C^0. For plate elements, the curvatures given by the second order derivatives take the role of displacements (comp. chapter 7). Hence the displacement function of a plate should assure continuity both of the surfaces of a plate deflection and its first derivations inside and on the boundaries between elements. Then the displacement field should be continuous and smooth within the plate. These functions are said to be of class C^1.

Criteria (A) and (B) have to be realised, criterion (C) does not. For instance, the shape function of plate elements does not often achieve the condition of continuity (continuity of the first derivations on boundaries of elements). If all criteria are realised, then we say that the described elements are ‘adjust ones’ If only criteria (A) and (B) are achieved, then elements are called ‘not adjust ones’.

The result of applying ‘adjust’ and ‘not adjust’ elements to discretization of a structure is presented inFigure 6. The convergence of results obtained with the help of the different types of elements which are used for discretization of a quadratic plate is shown in the same figure.

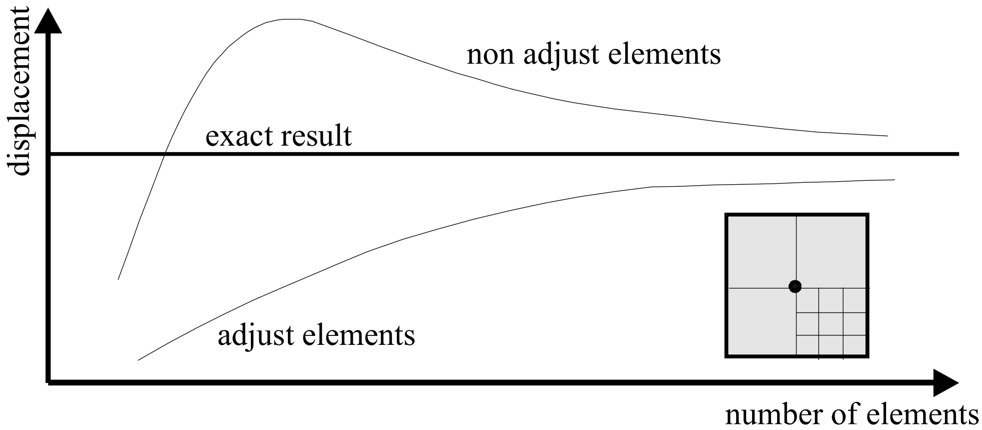


Figure 6. The precision of calculations for incompatible and non-incompatible elements depending on the number of elements.

Apart from the three listed criteria we can also add some others which determine the choice of approximation polynomials. This choice should assure isotropy with respect to axes of a coordinate system. We will show this using the example of building shape functions of plate elements (two- and three-dimensional problems). If we present approximation polynomials in the form of Pascal's triangle, then the choice of part of this triangle should be symmetrical in with respect to its axes. It is shown inFigure 7.

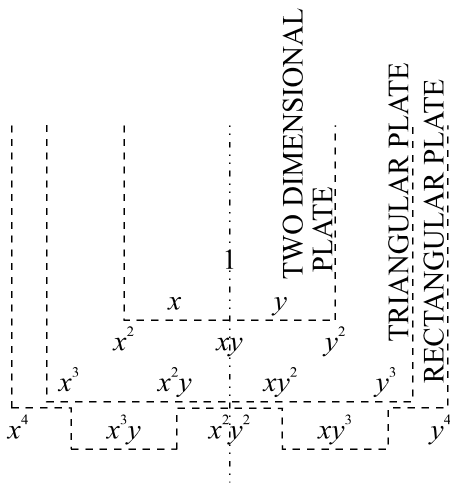


Figure 7. Pascal's triangle.

We can use Hermitte (described in chapter 4 of this book) and Lagrange polynomials (Zienkiewicz (1972)), but we always have to maintain the condition of isotropy.

There is a long list of references as far as shape functions are concerned but we recommend the following books: Bathe (1996), Rakowski and Kacprzyk (1993), Rao (1982), Zienkiewicz and Taylor (1994).

2D truss structures

2D trusses are one of the most common types of structures. The structure of a truss is economic since the ratio of the structure weight to forces carried by this structure is expressed as a small number. According to assumptions, loads (concentrated forces) will act on nodes only (temperature loads are an exception here) and connection bars will be joined with nodes in an articulated way. Although most structures which have been built lately are trusses with rigid nodes (they are basically frame constructions which are presented in Chapter 4), methods of solving problems in truss statics with articulated joints are still very important in engineering practice. The system of a plane truss with an articulated joint is the simplest example of an construction showing the idea of the finite element method without employing any complicated details. Though the structure of the method is very simple, most notions, algorithms and relations connected with the FEM algorithm will be relevant in discussions of more complex structures.

1.4. Basic relations and notations

We assume that the bar of a plane truss (we will also call it an element) is straight and homogeneous (it means that it is made from a homogeneous material without fractures and holes and has a constant cross section) and it joins nodes i (the first node) and j (the last node). Notations for these nodes (i, j) are local notations which are the same for all bars and they are to define element orientation. On the other hand, structure nodes also have global numbers which allow us to identify them. Global numbers are marked as n_i (the global number of the first node) and n_j (the global number of the last node). The node of a plane truss can move on the plane XY only. In mechanics, it means that the node has two degrees of freedom because in order to determine its location during its motion it should be given two coordinates. The situation of the node i of a rigid structure will be determined by initial coordinates X_i, Y_i with respect to the coordinate system which will be used for the description of the whole structure. We say that this system is global and its axes will be denoted by capital letters X, Y. The location of the node i, after its deformation caused by loads, is determined by two components of the displacement vector of nodes u_{iX} and u_{iY}. This method of description of the structure movement is called the Langrange description in mechanics. The description of some dependence between forces and element displacements becomes much simpler when we introduce a local coordinate system which will be denoted by small letters x, y. The x axis of the system overlaps the axis of the bar and has its beginning at the first node of an element i, while the y axis is perpendicular to the x axis and is directed in such a way that the Z axis of the global coordinate system and z axis of the local system have the same sense and direction. Because we accept that both coordinate systems are right-torsion, we can obtain the axis y by rotating the x axis clockwise through the angle $\pi/2$.

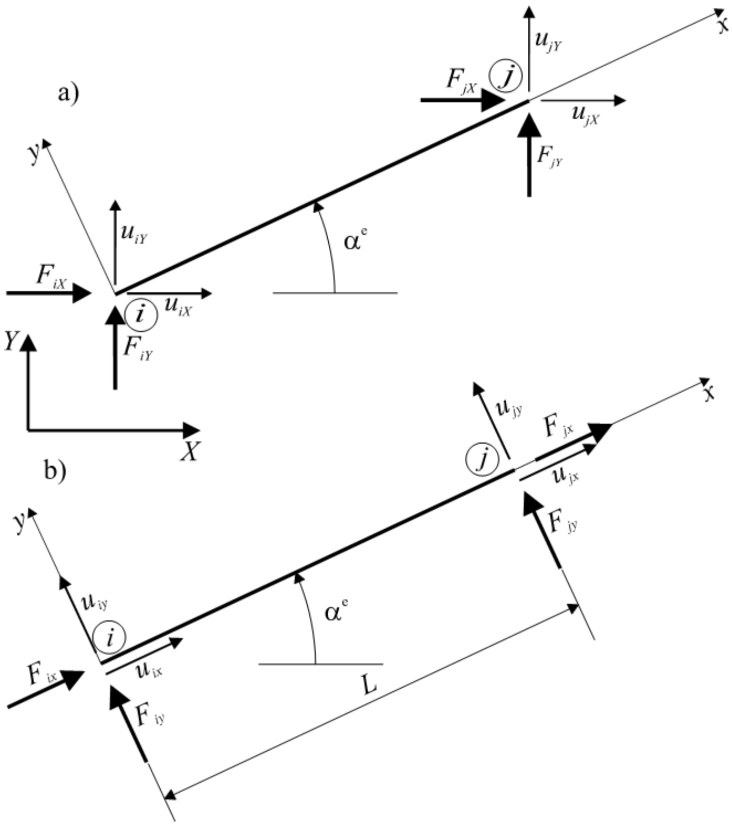


Figure 8.Nodal forces and displacements for the 2D truss element: a) in the global coordinate system; b) in the local coordinate system.

The most important notations, directions as well as senses of vectors and the coordinate systems are shown in Figure 8.

Nodal displacements and forces of elements are written as column matrices which we will call vectors.

The nodal displacement vector of the first node i and the last node j in the local coordinate system:

$$\mathbf{u}'_i = \begin{bmatrix} u_{ix} \\ u_{iy} \end{bmatrix}, \ \mathbf{u}'_j = \begin{bmatrix} u_{jx} \\ u_{jy} \end{bmatrix}. \tag{62}$$

The nodal displacement vector of the element *e* in the local coordinate system:

$$\mathbf{u}'^{e} = \begin{bmatrix} \mathbf{u}'_i \\ \mathbf{u}'_j \end{bmatrix} = \begin{bmatrix} u_{ix} \\ u_{iy} \\ u_{jx} \\ u_{jy} \end{bmatrix} \tag{63}$$

The nodal forces vector of the first node *i* and the last node *j* in the local coordinate system:

$$\mathbf{f'}_i = \begin{bmatrix} F_{ix} \\ F_{iy} \end{bmatrix}, \quad \mathbf{f'}_j = \begin{bmatrix} F_{jx} \\ F_{jy} \end{bmatrix} \tag{64}$$

The nodal forces vector of the element *e* in the local coordinate system:

$$\mathbf{f'}^{e} = \begin{bmatrix} \mathbf{f'}_i \\ \mathbf{f'}_j \end{bmatrix} = \begin{bmatrix} F_{ix} \\ F_{iy} \\ F_{jx} \\ F_{jy} \end{bmatrix}. \tag{65}$$

1.5. The element stiffness matrix of a plane truss in the local coordinate system

We look for the relation between nodal force vectors and nodal displacement vectors (comp. Chapter 1), which is necessary to express equilibrium equations depending on the nodal displacements

$$\mathbf{K'}^{e}\,\mathbf{u'}^{e} = \mathbf{f'}^{e}. \tag{66}$$

The general method of building such a relationship consists of using the principle of virtual work (comp. Chapter 1), but in this case we will apply different approach. We will use the equilibrium conditions in their basic forms which is possible in the case of bar elements.

Equilibrium equations for the element *e* (Figure 8) lead to the following relations:

$$\begin{aligned} &\sum F_x = F_{ix} + F_{jx} = 0 \\ &\sum F_y = F_{iy} + F_{jy} = 0 \\ &\sum M_i = F_{jy}\, L = 0 \end{aligned} \tag{67}$$

and we obtain

$$F_{iy} = 0\,; \quad F_{jy} = 0\,; \quad F_{ix} = -F_{jx}\,. \tag{68}$$

Since the set of three equilibrium Eqn. (67) or (68) contains four unknown parameters, this problem is statically indeterminate. The arrangement of an additional equation is necessary in order to make the determination of nodal forces possible. This equation ought to use the relation between nodal displacements of an element and its internal forces. Hooke's law written for a simple case of axial tension of a straight and homogeneous bar contains these relations (Figure 9):

$$\Delta L = \frac{N\,L}{E\,A}\,, \tag{69}$$

where N is the axial force in the bar (the positive value of an axial force always means tension), L is the bar length, ΔL signifies increment of the bar length due to the bar tension caused by the force N; E is Young's modulus of the material from which the bar is made; A is the area of the bar cross section.

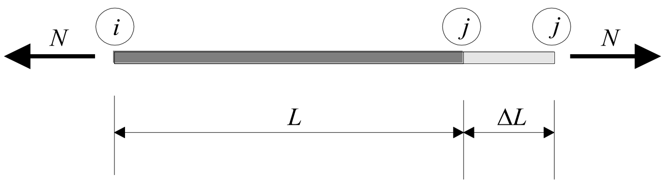


Figure 9. The bar streched along its own axis with the notation used in Eqn. (69).

Comparing Figure 8 and Figure 9we can observe simple relations between nodal forces acting on the bar, that is, F_{ix}, F_{jx} (Figure 8) and the axial force N (Figure 9):

$$F_{ix} = -N \;; \quad F_{jx} = N\,. \tag{70}$$

As it is shown above, these relations satisfy the third equilibrium Eqn. (68) identically.

The increment of the bar length due to tension results from axial displacements of the bar endings:

$$\Delta L = u_{jx} - u_{ix}\,, \tag{71}$$

which after inserting into Eqn. (69) leads to the relation:

$$N = \frac{EA}{L}\left(u_{jx} - u_{ix}\right). \tag{72}$$

Taking into consideration the relationship between the axial force of the element and nodal forces Eqn. (70) with respect to Eqn. (72) we obtain:

$$F_{ix} = \frac{EA}{L}\left(u_{ix} - u_{jx}\right);\; F_{jx} = \frac{EA}{L}\left(-u_{ix} + u_{jx}\right) \tag{73a}$$

$$F_{iy} = 0;\quad F_{jy} = 0. \tag{73b}$$

The resulting relations are the searched relations Eqn. (66) between the nodal forces and nodal displacements of the truss element. We will write them one more time in a different form:

$$\begin{bmatrix} \frac{EA}{L} & 0 & -\frac{EA}{L} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{EA}{L} & 0 & \frac{EA}{L} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{ix} \\ u_{iy} \\ u_{jx} \\ u_{jy} \end{bmatrix} = \begin{bmatrix} F_{ix} \\ F_{iy} \\ F_{jx} \\ F_{jy} \end{bmatrix}. \tag{74}$$

After considering notations Eqn. (63), (65) and (66), the above form leads to the equation:

$$\mathbf{K}'^{e} = \begin{bmatrix} \frac{EA}{L} & 0 & -\frac{EA}{L} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{EA}{L} & 0 & \frac{EA}{L} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{75}$$

which defines a matrix $\mathbf{K}'^{e}$. This matrix will be called the element stiffness matrix of a plane truss. The matrix in the form of Eqn. (75) expresses relationships between the vector $\mathbf{u}'^{e}$ and the nodal force vector of an element $\mathbf{f}'^{e}$ in the local coordinate system.

The stiffness matrix $\mathbf{K}'^{e}$ can be simplified to:

$$\mathbf{K}'^{e} = \begin{bmatrix} \mathbf{J}' & -\mathbf{J}' \\ -\mathbf{J}' & \mathbf{J}' \end{bmatrix}, \tag{76}$$

where $\mathbf{J}'$ is the square matrix defined in the following way:

$$\mathbf{J}' = \frac{EA}{L}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{77}$$

1.6. Coordinate system rotation

The form of the element stiffness matrix determined in the local coordinate system will not be convenient in further considerations for which we will use matrices of different elements. The most convenient method is transforming all matrices to the form which is defined in one common coordinate system. Such a system will be called *the global coordinate system*. It can be the system of a certain type: cartesian, polar or curvilinear. The cartesian coordinate system is the most convenient system for a truss. Nodal coordinates of a structure are usually given in the global coordinate system.

Now we convert the element stiffness matrix to the global system. We start the transformations by finding relationships for a single node:

$$\begin{aligned} u_{iX} &= u_{ix}\cos\alpha - u_{iy}\sin\alpha \\ u_{iY} &= u_{ix}\sin\alpha + u_{iy}\cos\alpha \end{aligned} \tag{78}$$

or in matrix form:

$$\begin{bmatrix} u_{iX} \\ u_{iY} \end{bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}\begin{bmatrix} u_{ix} \\ u_{iy} \end{bmatrix}, \tag{79}$$

where $c = \cos\alpha$ and $s = \sin\alpha$.

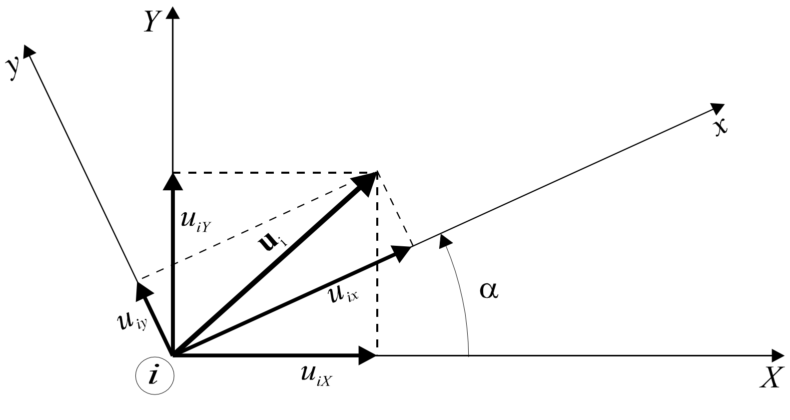


Figure 10. Displacement vector components in the global and local coordinate systems rotated through the α angle.

Denoting

$$\mathbf{u}_i = \begin{bmatrix} u_{iX} \\ u_{iY} \end{bmatrix} \tag{80}$$

and taking into consideration notation Eqn. (62), we obtain:

$$\mathbf{u}_i = \mathbf{R}_i \mathbf{u}'_i, \tag{81}$$

where $\mathbf{R}_i = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$ (82)

is the transformation matrix of the vector $\mathbf{u}'_i$ from the local system to the global one.

A reverse relation will be required:

$$\mathbf{u}'_i = \left(\mathbf{R}_i\right)^{-1} \mathbf{u}_i, \tag{83}$$

where $\left(\mathbf{R}_i\right)^{-1}$ is the inverse matrix of $\mathbf{R}_i$; it means that it has such a property that

$$\mathbf{R}_i \left(\mathbf{R}_i\right)^{-1} = \mathbf{I}, \tag{84}$$

where **I** is the identity matrix

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{85}$$

The matrix $\mathbf{R}_i$ like other transformation matrices has the property that

$$\left(\mathbf{R}_i\right)^{-1} = \left(\mathbf{R}_i\right)^{\mathsf{T}}, \tag{86}$$

it means that $\mathbf{R}_i$ is the orthogonality matrix (the determinant of this matrix is equal to 1, i.e. $\det(\mathbf{R}_i)=1$; $\det\left(\mathbf{R}_i\right)^{\mathsf{T}} = 1$). We can easily check the property Eqn. (86) of the matrix $\mathbf{R}_i$ by making a direct calculation:

$$\mathbf{R}_i \left(\mathbf{R}_i\right)^{\mathsf{T}} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \cdot \begin{bmatrix} c & s \\ -s & c \end{bmatrix} = \begin{bmatrix} c^2+s^2 & cs-sc \\ sc-cs & c^2+s^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}.$$

The transformation matrix contains the blocks of the nodal transformation matrix:

$$\mathbf{R}^e = \begin{bmatrix} \mathbf{R}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_j \end{bmatrix}, \tag{87}$$

where $\mathbf{R}_i$ is the transformation matrix of the first node, $\mathbf{R}_j$ is the transformation matrix of the last node and **0** is the part of the matrix containing zero values. The transformation matrices $\mathbf{R}_i$ and $\mathbf{R}_j$ are usually identical (for straight elements) because rotation angles of the vector of nodes i and j are equal. Since the truss elements are straight, we can write $\mathbf{R}_i = \mathbf{R}_j$.

Finally, the relationships between the nodal displacement vector of the element expressed in the local system and the same vector in the global system have the form:

$$\mathbf{u}^e = \mathbf{R}^e \mathbf{u}'^e \tag{88}$$

$$\mathbf{u}'^e = \left(\mathbf{R}^e\right)^{\mathsf{T}} \mathbf{u}^e \tag{89}$$

The relationship between the nodal force vector of an element in the local system and the same vector in the global system is identical to the relationship that we have obtained in the equations describing displacements

$$\mathbf{f}_i = \mathbf{R}_i \mathbf{f}'_i \tag{90}$$

and

$$\mathbf{f}'_i = \left(\mathbf{R}_i\right)^{\mathsf{T}} \mathbf{f}_i \,, \tag{91}$$

$$\mathbf{f}^e = \mathbf{R}^e \mathbf{f}'^e \,, \tag{92}$$

$$\mathbf{f}'^e = \left(\mathbf{R}^e\right)^{\mathsf{T}} \mathbf{f}^e \,. \tag{93}$$

1.7. The element stiffness matrix in the global coordinate system

Multiplying Eqn. (66) by the transformation of the matrix $\mathbf{R}^e$ and substituting relation (89) for $\mathbf{u}'^e$, we obtain

$$\mathbf{R}^e \mathbf{K}'^e \left(\mathbf{R}^e\right)^{\mathsf{T}} \mathbf{u}^e = \mathbf{R}^e \mathbf{f}'^e \tag{94}$$

On the basis of relation Eqn. (92) the right hand side of this equation is equal to $\mathbf{f}^e$, so if we introduce the notation

$$\mathbf{K}^e = \mathbf{R}^e \mathbf{K}'^e \left(\mathbf{R}^e\right)^{\mathsf{T}} \tag{95}$$

we obtain

$$\mathbf{f}^e = \mathbf{K}^e \mathbf{u}^e \,, \tag{96}$$

It is the required relationship between nodal forces and displacements of the element in the global coordinate system.

If we perform the multiplication in Eqn. (95), we obtain

$$\mathbf{K}^e = \begin{bmatrix} \mathbf{J} & -\mathbf{J} \\ -\mathbf{J} & \mathbf{J} \end{bmatrix}, \tag{97}$$

where $\mathbf{J} = \frac{EA}{L}\begin{bmatrix} c^2 & sc \\ sc & s^2 \end{bmatrix}.$ (98)

We can exchange form Eqn. (98) of the matrix **J** into the equivalent one in which trigonometric functions do not exist. Let us note that

$$c = \cos\alpha = \frac{L_X}{L} \quad \text{and} \quad s = \sin\alpha = \frac{L_Y}{L}. \tag{99}$$

After inserting these relations into Eqn. (98), we obtain

$$\mathbf{J} = \frac{EA}{L^3}\begin{bmatrix} L_X^2 & L_X L_Y \\ L_X L_Y & L_Y^2 \end{bmatrix} \tag{100}$$

1.8. Nodal equilibrium equations and aggregation of a stiffness matrix

Replacing existing bars (elements) of a truss by nodal forces we obtain a group of nodes which can be treated as material particles with two degrees of freedom. These nodes are loaded with concentrated forces coming from elements or external loads. The equilibrium conditions for such a node are as follows:

$$\begin{aligned} \sum P_X &= \sum_{k=1}^{E_n}\left(-F_{nX}^{e_k}\right) + P_{nX} = 0, \\ \sum P_Y &= \sum_{k=1}^{E_n}\left(-F_{nY}^{e_k}\right) + P_{nY} = 0, \end{aligned} \tag{101}$$

where we have denoted $F_{nX}^{e_k}$ - component in the direction X of nodal forces from the element numbered e_k acting on a node n, P_{nX} - component in the direction X of the external forces acting on the node n, E_n - number of elements joined to the node n.

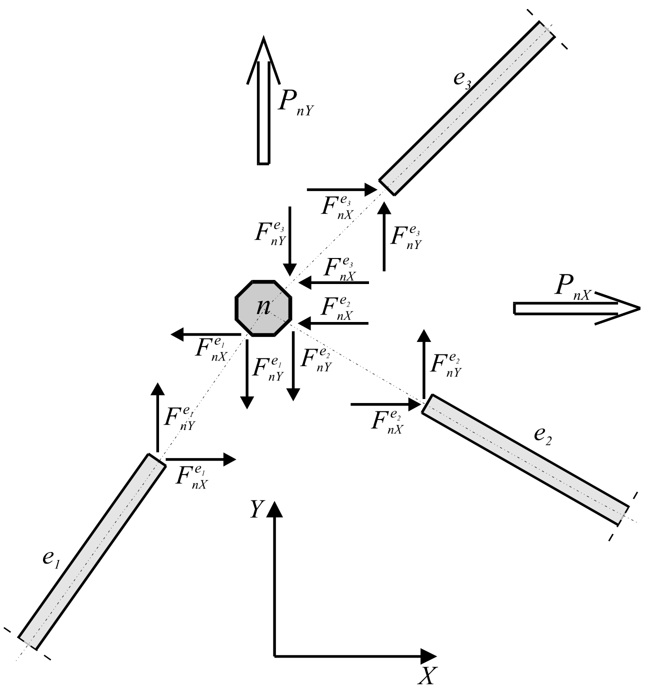


Figure 11. Nodal and external forces acting on the truss node.

Now we transform the set of Eqn. (101) to the form containing nodal displacements:

$$\begin{bmatrix} \mathbf{K}_{1n} & \mathbf{K}_{2n} & \dots & \mathbf{K}_{in} & \dots & \mathbf{K}_{N_n n} \end{bmatrix} \mathbf{u} = \mathbf{p}_n \qquad (102)$$

In Eqn. (102)

$$\mathbf{u} = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_i \\ \vdots \\ \mathbf{u}_{N_n} \end{bmatrix}$$ signifies the global vector of nodal displacements of a structure,

$$\mathbf{p}_n = \begin{bmatrix} P_{nX} \\ P_{nY} \end{bmatrix}$$ is the vector of external forces acting on the node *n*,

matrices $\mathbf{K}_{in}$ are quadratic matrices with dimensions 2x2 determined as follows:

where i=n - $\mathbf{K}_{nn} = \sum_{k=1}^{E_n} \mathbf{J}^{e_k}$, (103)

$e_1, e_2 \ldots e_k \ldots e_{E_n}$ - are numbers of the elements joined at node *n*,

if $i \neq n$ and nodes *i* and *n* are not directly connected by any elements, then $\mathbf{K}_{in} = 0$,

if $i \neq n$ and nodes *i* and *n* are connected by some element with a number *e*, then $\mathbf{K}_{in} = -\mathbf{J}^e$,

$\mathbf{J}^e$ – signifies the block of a stiffness matrix of the element *e* (comp. Eqn. (98)).

Arranging equilibrium Eqn. (102) for all nodes of a structure we obtain the final form of equations allowing determination of nodal displacements of the truss:

$$\left[\begin{array}{c:c:c:c:c:c} \mathbf{K}_{11} & \mathbf{K}_{12} & \cdots & \mathbf{K}_{1n} & \cdots & \mathbf{K}_{1N_n} \\ \hdashline \mathbf{K}_{21} & \mathbf{K}_{22} & \cdots & \mathbf{K}_{2n} & \cdots & \mathbf{K}_{2N_n} \\ \hdashline \vdots & \vdots & \ddots & \vdots & & \vdots \\ \hdashline \mathbf{K}_{n1} & \mathbf{K}_{n2} & \cdots & \mathbf{K}_{nn} & \cdots & \mathbf{K}_{nN_n} \\ \hdashline \vdots & \vdots & & \vdots & \ddots & \vdots \\ \hdashline \mathbf{K}_{N_n1} & \mathbf{K}_{N_n2} & \cdots & \mathbf{K}_{N_nn} & \cdots & \mathbf{K}_{N_nN_n} \end{array}\right] \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_n \\ \vdots \\ \mathbf{u}_{N_n} \end{bmatrix} = \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \vdots \\ \mathbf{p}_n \\ \vdots \\ \mathbf{p}_{N_n} \end{bmatrix}$$

or $\mathbf{Ku} = \mathbf{p}$ (104)

The matrix **K** of the set of Eqn. (104) is the global stiffness matrix of the structure, the vector **u** is the global vector of nodal displacements of the structure and the vector **p** is the global vector of nodal forces of the structure.

Careful numbering of the nodes can allow **K** to the banded matrix which is characterised by a fact that non-zero components appear on the main diagonal and closely to it. The matrix **K** is a symmetric matrix which means that its components satisfy equations:

$$K_{ij} = K_{ji} \text{ or } \mathbf{K} = \mathbf{K}^{\mathsf{T}} \tag{105}$$

which result from the principle of virtual work (comp. Chapter 1). Components K_{nn} which are on the main diagonal are always positive

$$K_{nn} > 0 \tag{106}$$

which is a direct conclusion drawn from definitions Eqn. (98) and(103).

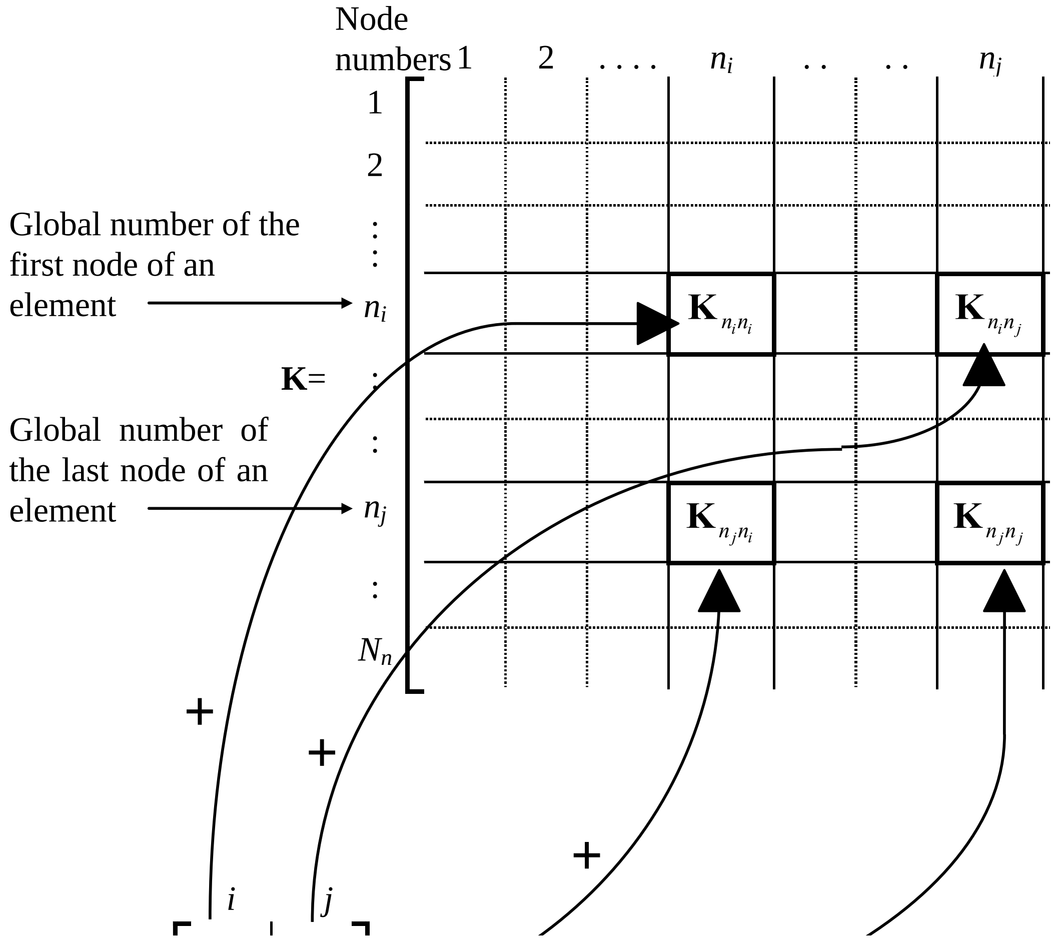


Figure 12. The stiffness matrix aggregation scheme.

The zero component K_{nn} demonstrates geometric changability of a structure and should be removed by a suitable change of a geometric scheme. The matrix **K** presented in Eqn. (104) is a singular matrix (it means det **K**=0), hence the set of Eqn. (104) cannot be solved without modifying it. This modification will depend on the consideration of boundary conditions. We will consider this problem in the next section.

The process of building the global stiffness matrix is called aggregation of a matrix. It can be done by means of the method described in Chapter 1 demanding formation of connection matrices. Since these matrices are large, then their use is not convenient and they are rarely used in computer implementation of the FEM algorithm. The method of summation of blocks shown by Eqn. (102) and (103) is much simpler. The form of matrix Eqn.(102) and (103) may seem to be complicated, but in fact, we have very simple operations of insertion of blocks here. This method is best shown in Figure 12.

'+' signs located at arrows pointing to the place of location of blocks $\mathbf{K}^e$ mean that blocks $\mathbf{J}^e$ should be added to the existing contents of 'cells' of matrices $\mathbf{K}_{n_i n_i}$ or $\mathbf{K}_{n_j n_j}$, and blocks $-\mathbf{J}^e$ lying beyond the diagonal should be added to 'cells' $\mathbf{K}_{n_i n_j}$ or $\mathbf{K}_{n_j n_i}$. In the case of a truss where nodes are usually joined by one element, blocks lying beyond the main diagonal contain only a single matrix $-\mathbf{J}^e$. But blocks lying on the main diagonal $\mathbf{K}_{n_i n_i}$ contain sums of as many matrices $\mathbf{J}^e$ as elements joined with the node n_i.

1.9. Boundary conditions

As it was noted in the previous section of this Chapter that the global stiffness matrix of a structure is most often a singular matrix directly after the aggregation. It means that the determinant of this matrix is equal to zero. Because the set of Eqn. (104) has to have only one solution for static problems, we have to modify the global stiffness matrix. It should be done in such a way that the solution of the set of linear Eqn. (104) is possible. The reason for the singularity of the matrix **K** is the lack of information about supports of the construction, thus we need to define what the support of the node is.

For trusses there are two types of supports possible: an articulated support and an articulated movable support. The articulated support (shown in Figure 13a) prevents movements of a node in any direction which means:

$$u_{rX} = 0, \; u_{rY} = 0. \tag{107}$$

The movement of the support node r causes reactions in two components: R_X and R_Y (Figure 13a), which counteract the movement of the node r. We say that this support assures *free support* of a node.

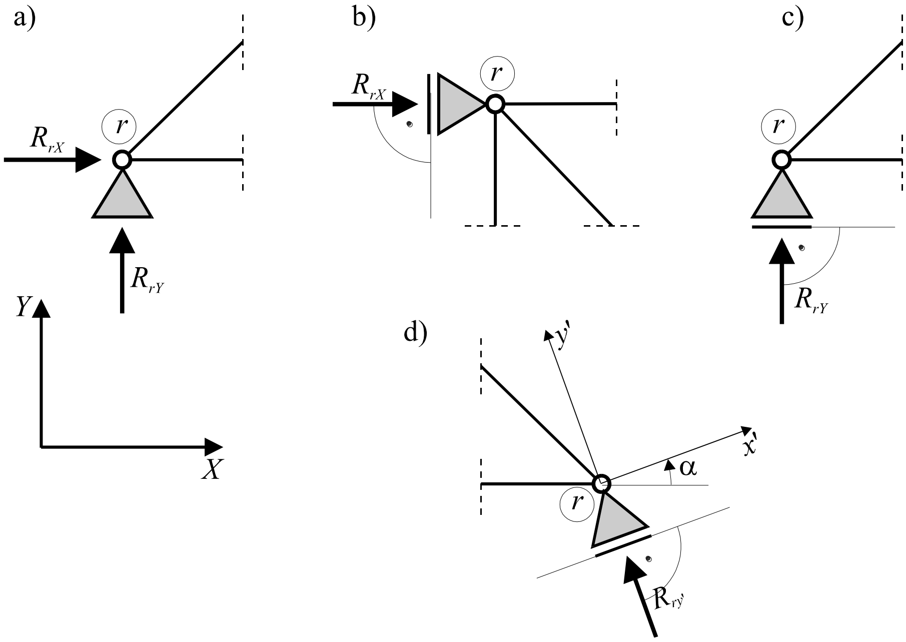


Figure 13. Plane truss support types.

The next support shown in Figure 13b is called an articulated movable support and it prevents movements of a node along one line only, but it allows movement of a node in perpendicular direction with respect to this line. The reaction occurring in the articulated movable support can have the direction of this line only (Figure 13b,c,d). The support can appear in a few forms, two most often occurring variants (shown in Figure 13b,c) give very simple support conditions:

- support with the possibility of movement along the Y axis of the global coordinate system (Figure 13b)

$$\mathbf{u}_{rX} = 0, \tag{108}$$

- support with the possibility of movement along the X axis of the global coordinate system (Figure 13c)

$$\mathbf{u}_{rY} = 0. \tag{109}$$

The third variant of a movable support causes problems when describing the boundary conditions because the direction of the reaction of this support (Figure 13d) is not parallel to any axis of the global coordinate system. It is important because equilibrium Eqn. (101) leading to Eqn. (104) were written in the global coordinate system. In the case of a support with movement not parallel to any axis of the global coordinate system (we will call such supports skew supports) we have to write the boundary conditions in the system *x*'*y*' connected with the support. The system *x*'*y*' is rotated with respect to the global system by an angle α' (Figure 13d). We will explain the transformation method for a set of equations at a support node to the local system in the next section. Now we will focus on describing the boundary condition. We write the condition of absence of a movement along the *y*' axis analogously as in Eqn. (109):

$$\mathbf{u}_{ry'} = 0 \quad . \tag{110}$$

Eqn. ((107)...(110) describing the boundary conditions give us the values of displacements at support nodes. Hence some equations of set Eqn. (104) should be removed, because they contain unknown forces acting on support nodes (constraint reactions). These equations can be replaced by equations of boundary conditions (for example Eqn. (107)). It is usually done by modifying some Eqn. (104).

Let *m* be the global number of the degree of freedom which is eliminated by the boundary condition: $u_m = 0$, then we modify the row with the number *m* in the global stiffness matrix **K**, replacing it by a row containing zeros and the value 1 in the column *m*:

$$\begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} & \cdots & \mathbf{K}_{1m} & \cdots & \mathbf{K}_{1N_n} \\ \mathbf{K}_{21} & \mathbf{K}_{22} & \cdots & \mathbf{K}_{2m} & \cdots & \mathbf{K}_{2N_n} \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ \mathbf{K}_{N_n1} & \mathbf{K}_{N_n2} & \cdots & \mathbf{K}_{N_nm} & \cdots & \mathbf{K}_{N_nN_n} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ \mathrm{u_m} \\ \vdots \\ u_{N_n} \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ 0 \\ \vdots \\ P_{N_n} \end{bmatrix}$$

or $\mathbf{K}^o \mathbf{u} = \mathbf{p}^r$. (111)

The nodal load vector **p** should be modified so that equation *m* contains zero on the right side. The modified matrices are marked in Eqn. (111) by a superscript *r*.

These changes in the stiffness matrix disturb the symmetry because $K_{im} \neq 0$ but $K_{im} = 0$ when $i \neq m$ (comp. Eqn. (111)). The absence of symmetry in the stiffness matrix does not prevent the solving of the equilibrium Eqn. (104) but it considerably loads the computer memory storing coefficients K_{ij} either in the core memory (RAM) or external space (disk) which lengthens the solution time for a set of equations (comp. Appendix 2). Thus, let us try to restore the symmetry of the matrix $\mathbf{K}^o$ (Eqn. (111)). Let us note that the terms located in the column with the number m are multiplied by the zero value of the displacement u_m. Hence we can insert zeros instead of coefficients in the column m (except for one coefficient in the row m which has to be equal to 1). If we modify the stiffness matrix in that way, the solution of our problem will be the same and the matrix will be a symmetric one:

$$\mathbf{K}^r = \left[\begin{array}{c:c:c:c:c:c} \mathbf{K}_{11} & \mathbf{K}_{12} & \cdots & 0 & \cdots & \mathbf{K}_{1N_n} \\ \hdashline \mathbf{K}_{21} & \mathbf{K}_{22} & \cdots & 0 & \cdots & \mathbf{K}_{2N_n} \\ \hdashline \vdots & \vdots & \ddots & \vdots & & \vdots \\ \hdashline 0 & 0 & \cdots & 1 & \cdots & 0 \\ \hdashline \vdots & \vdots & & \vdots & \ddots & \vdots \\ \hdashline \mathbf{K}_{N_n1} & \mathbf{K}_{N_n2} & \cdots & 0 & \cdots & \mathbf{K}_{N_nN_n} \end{array}\right] \tag{112}$$

Finally, we solve the problem:

$$\mathbf{K}^r \mathbf{u} = \mathbf{p}^r, \tag{113}$$

where the matrix $\mathbf{K}^r$ is symmetrical and is not singular which means that $\det \mathbf{K}^r \neq 0$, if we have properly chosen the boundary conditions. On the basis of the theorem about the positive value of a strain energy (comp. Eqn. (45), Chapter 1) we can conclude that the matrix $\mathbf{K}^r$ has to be positive-determinant, then

$$\det \mathbf{K}^r > 0. \tag{114}$$

Hence the set of Eqn. (113) has one solution.

In small finite element systems (programs) the matrix $\mathbf{K}^r$ is usually left in the form noted in Eqn. (112). Large and complex systems used to solve problems described by many thousands of equations usually remove rows and columns containing zeros from the matrix $\mathbf{K}^r$ and vector $\mathbf{p}^r$. This is done to reduce the dimensions of a solved problem. This method of modification of the matrix $\mathbf{K}^r$ requires re-numbering of degrees of freedom of a structure. Because it is not strictly joined with FEM and is connected with the computer implementation of the FEM algorithm, we will not describe it here.

1.10. Transformation of the stiffness matrix for a 'skew' support

Now we are explaining ways of transforming an element stiffness matrix joined to a support node by means of a 'skew' support (Figure 13d). We choose the coordinate system *x*'*y*' in such a way that the direction of a support reaction covers the *y*' axis and the movement will be parallel to the *x*' axis (an alternative choice of the local coordinate system is obviously possible). The *x*' axis is rotated with respect to the *X* axis of the global system by the angle α' which we will deem to be positive when the rotation from the *X* axis to the *x*' axis is anticlockwise. The positive angle α' is shown in Figure 13d.

If we write equilibrium equations for the support node *r* in the system *x*'*y*', then the boundary condition of this support is determined by Eqn. (110). Let us try to perform the necessary transformation. We make use of relations Eqn. (81) and (83) which served us in Sec. 2.3 to pass from the local system of an element to the global one.

Then we express the nodal forces vector at the node *r* as follows:

$$\begin{bmatrix} F_{rx'} \\ F_{ry'} \end{bmatrix} = \begin{bmatrix} c' & s' \\ -s' & c' \end{bmatrix} \begin{bmatrix} F_{rX} \\ F_{rY} \end{bmatrix},$$

or in an abbreviated form:

$$\mathbf{f}'_r = \left(\mathbf{R}'_r\right)^{\mathrm{T}} \mathbf{f}_r \,. \tag{115}$$

Next we transform the nodal displacements vector of the support node from the local system to the global one as follows:

$$\begin{bmatrix} u_{rX} \\ u_{rY} \end{bmatrix} = \begin{bmatrix} c' & -s' \\ s' & c' \end{bmatrix} \begin{bmatrix} u_{rx'} \\ u_{ry'} \end{bmatrix},$$

or in a close form:

$$\mathbf{u}_r = \mathbf{R}'_r \mathbf{u}'_r \,. \tag{116}$$

In Eqn. (115) and (116) we have marked

$$\mathbf{R}'_r = \begin{bmatrix} c' & -s' \\ s' & c' \end{bmatrix}, \; c' = \cos\alpha', \; s' = \sin\alpha'$$

and $(\mathbf{R}'_r)^{\mathrm{T}}$ is the transpose of the matrix $\mathbf{R}'_r$.

Let us assume that an element e joins nodes r_i and r_j supported by 'skew' supports which are rotated by angles α'_i and α'_j (Figure 14). Then we write equilibrium equations for nodes r_i and r_j in the local coordinate system $x'_i y'_i$ at the node r_i and $x'_j y'_j$ at the node r_j. The transformation of nodal forces vectors and nodal displacements vectors of the element e is as follows:

- for a nodal forces vector

$$\mathbf{f}'^{e} = \left(\mathbf{R}'^{e}\right)^{\mathrm{T}} \mathbf{f}^{e} \tag{117}$$

or in a developed form

$$\begin{bmatrix} \mathbf{f}'_{r_i} \\ \mathbf{f}'_{r_j} \end{bmatrix} = \begin{bmatrix} \left(\mathbf{R}'_{r_i}\right)^{\mathrm{T}} & \mathbf{0} \\ \mathbf{0} & \left(\mathbf{R}'_{r_j}\right)^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{f}_{r_i} \\ \mathbf{f}_{r_j} \end{bmatrix},$$

- for the nodal displacements vector

$$\mathbf{u}^{e} = \mathbf{R}'^{e} \, \mathbf{u}'^{e} \tag{118}$$

or $\begin{bmatrix} \mathbf{u}_{r_i} \\ \mathbf{u}_{r_j} \end{bmatrix} = \begin{bmatrix} \mathbf{R}'_{r_i} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}'_{r_j} \end{bmatrix} \begin{bmatrix} \mathbf{u}'_{r_i} \\ \mathbf{u}'_{r_j} \end{bmatrix}.$

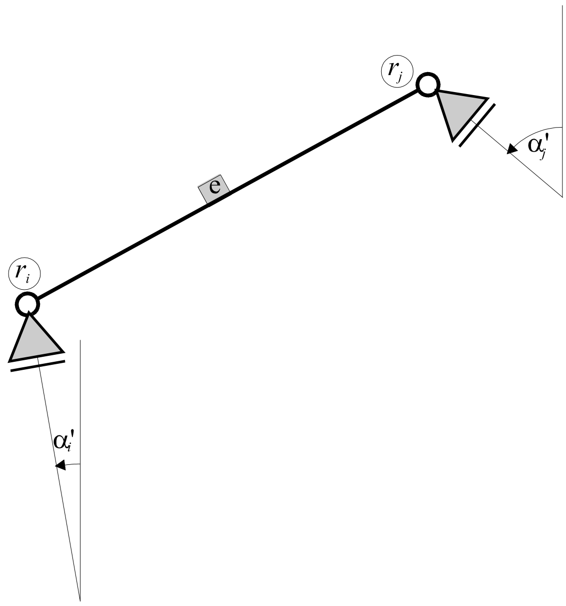


Figure 14. The bar with 'skew' supports.

Inserting relationship Eqn. (118) into (96) and the result into(117), we get the equation transforming the stiffness matrix of the element *e* from the global coordinate system to the support coordinate system:

$$\mathbf{f}'^{e} = \left(\mathbf{R}'^{e}\right)^{\mathsf{T}} \mathbf{K}^{e} \mathbf{R}'^{e} \mathbf{u}'^{e} \tag{119}$$

We simplify this equation to the form:

$$\mathbf{f}'^{e} = \mathbf{K}'^{e} \mathbf{u}'^{e}, \tag{120}$$

in which we make use of the substitution:

$$\mathbf{K}'^{e} = \left(\mathbf{R}'^{e}\right)^{\mathsf{T}} \mathbf{K}^{e} \mathbf{R}'^{e}, \tag{121}$$

defining the element matrix in the support coordinate system.

One of angles α' (Figure 14) is most often equal to zero because it rarely happens that a truss bar joins two support nodes supported by a 'skew' support. The transformation matrix of a zero angle is a unit matrix. Because (c'=1, s'=0), then the element transformation matrix is simplified to the form:

$$\mathbf{R}'^{e} = \begin{bmatrix} \mathbf{R}'_{r_i} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \tag{122}$$

when the second node is described in the global system but we transform forces and displacements at the first node r_i, and

$$\mathbf{R'}^e = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{R'}_{r_j} \end{bmatrix}, \tag{123}$$

when the transformation concerns the last node r_j only.

As it has been shown that the existence of ‘skew’ supports complicates the simple FEM algorithm presented in Chapter 1 because it requires additional transformations of element stiffness matrices before the aggregation of the global matrix is done. There are some other simpler, though approximate, methods of solving this problem and they will be discussed in the next section concerning boundary elements.

1.11. Elastic supports and boundary elements

Not all kinds of supports applied to support trusses can be described by the boundary conditions of types Eqn. (108), (109) and(110). There are flexible supports which have displacements connected with a support reaction, for instance, the linear relation of the following type:

$$\begin{aligned} R_{rX} &= -h_{rX} u_{rX}, \\ R_{rY} &= -h_{rY} u_{rY}, \end{aligned} \tag{124}$$

where h_{rX} is the support stiffness in the direction of the X axis and h_{rY} is the support stiffness in the direction of the Y axis. The linear spring shown in Figure 15is a good model of this type of support.

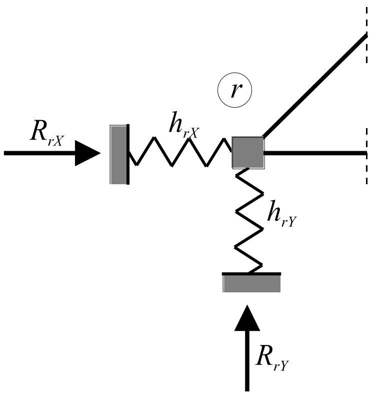


Figure 15. The elastic support model.

If we treat reactions R_{rX} and R_{rY} acting on the node supported elastically as external forces, then we obtain the nodal forces vector containing unknown displacements u_{rX}, u_{rY}:

$$\mathbf{p}=\left[\begin{array}{c} P_{1X}\\ P_{1Y}\\ \hline P_{2X}\\ P_{2Y}\\ \hline \vdots \\ \hline R_{rX}\\ R_{rY}\\ \hline \vdots\\ \hline P_{N_nX}\\ P_{N_nY}\end{array}\right]\begin{array}{l}1\\ \\ 2\\ \\ \vdots\\ r\\ \\ \vdots\\ N_n\\ \\ \end{array} = \left[\begin{array}{c} P_{1X}\\ P_{1Y}\\ P_{2X}\\ P_{2Y}\\ \vdots \\ -h_{rX}u_{rX}\\ -h_{rY}u_{rY}\\ \vdots\\ P_{N_nX}\\ P_{N_nY}\end{array}\right] \tag{125}$$

The vector **p** cannot be absolutely used as the right hand side of Eqn. (104) in which unknown values of nodal displacements should be on the left hand side of the equation. Now we are transforming the vector **p** described by Eqn. (125) in such a way that nodal reactions of the elastic node *r* will be moved to the left hand side of the equilibrium equation:

$$\mathbf{K}^s\mathbf{u} = \mathbf{p}^r, \tag{126}$$

where $\mathbf{K}^s$ is the stiffness matrix containing information about elastic supports of the structure and $\mathbf{p}^r$ is the nodal forces vector in which the boundary conditions written in Eqn. (111) (we can treat the elastic supports as fixed ones after transferring the relations which described them to the left hand side of the equation) are considered.

The matrix $\mathbf{K}^s$ is written by the equation:

$$\mathbf{K}^s = \left[\begin{array}{cc|c|cc|c|c}
\mathbf{K}_{11} & \mathbf{K}_{12} & & \mathbf{K}_{1m} & \mathbf{K}_{1(m+1)} & & \mathbf{K}_{1N_n} \\
\mathbf{K}_{21} & \mathbf{K}_{22} & \dots & \mathbf{K}_{2m} & \mathbf{K}_{2(m+1)} & \dots & \mathbf{K}_{2N_n} \\
\hline
& \vdots & \ddots & & \vdots & & \vdots \\
\hline
\mathbf{K}_{m1} & \mathbf{K}_{m2} & & \mathbf{K}_{mm} + h_{rX} & \mathbf{K}_{m(m+1)} & & \mathbf{K}_{mN_n} \\
\mathbf{K}_{(m+1)1} & \mathbf{K}_{(m+1)2} & \dots & \mathbf{K}_{(m+1)m} & \mathbf{K}_{(m+1)(m+1)} + h_{rY} & \dots & \mathbf{K}_{(m+1)N_n} \\
\hline
& \vdots & & & \vdots & \ddots & \vdots \\
\hline
\mathbf{K}_{N_n1} & \mathbf{K}_{N_n2} & \dots & \mathbf{K}_{N_nm} & \mathbf{K}_{N_n(m+1)} & \dots & \mathbf{K}_{N_nN_n}
\end{array}\right]
\begin{array}{c} 1 \\ \\ \vdots \\ r \\ \\ N_n \\ \\ \end{array} \qquad (127)$$

where m is the global number of the first degree of freedom of the node r. With standard numbering $m=(r-1)N_D+1$ where N_D is the number of degrees of freedom of the node. For a 2D truss $N_D=2$, the number of the first degree of freedom of the node r is equal to $m=2r-1$.

At this stage, the modified matrix $\mathbf{K}^s$ contains the stiffness of elastic supports which are added to the terms coming from the truss element of a structure. These sums are located on the main diagonal of the matrix in rows describing the equilibrium of the node r. Such an interpretation of elastic supports leads to a convenient, although simplistic, way of considering fixed supports. We substitute them for elastic supports with very large stiffness, for example $H=1\times10^{30}$ onto the main diagonal. This method was formulated by Irons and Ahmad (1980) who multiplies terms lying in a suitable row on the diagonal of the matrix **K** by numbers of the order of 10^6. After inserting a high value onto the diagonal, it is irrelevant to insert zeros both in rows and columns of the matrix **K** as well as rows of the vector of the right hand side **p**. It is very important for large stiffness matrices which are often stored in structures of data different from quadratic tables (comp. Appendix 2). The simplicity of this method ensures that it is commonly used in the computer implementation of the FEM algorithm instead of the exact method described in Sec.2.6.

Elastic supports also suggest the use of a special support element which could substitute any elastic constraints and fixed supports (which should be treated as elastic supports with large stiffness). This support element rotated by an angle α with respect to the global coordinate system is shown inFigure 16.

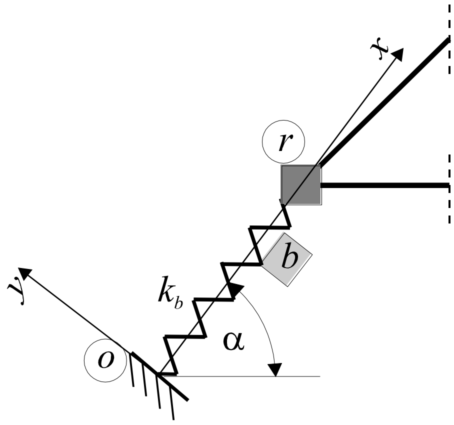


Figure 16The boundary element scheme.

We can easily obtain the stiffness matrix of such an element from the matrix of an ordinary truss element described by Eqn. (75) in the local coordinate system or Eqn. (97) in the global system. We do it in such a way that we substitute the stiffness of a bar EA/L for the stiffness of the elastic boundary element k_b. In general, the node o of this element is always fixed, so we can remove it from the set of equations which allows us to treat the boundary element as an element with two degrees of freedom:

$$\mathbf{K}^b = k_b \begin{bmatrix} c^2 & sc \\ sc & s^2 \end{bmatrix}, \tag{128}$$

where similarly to Eqn. (78) $c = \cos\alpha$, $s = \sin\alpha$.

When we want to substitute the fixed support for this element we accept k_b=H. The value of H depends on the computer system in which the program will be started and most of all it depends on the type of real numbers. We can take for example H=1×10^{30} as reference for many systems.

1.12. The nodal loads vector with temperature load

As we have already noted in the introduction to this Chapter, truss loads which act on elements and do not act on nodes directly are temperature loads. Now we will show how we can replace this load by known loads, that is, concentrated forces acting on the nodes of a structure.

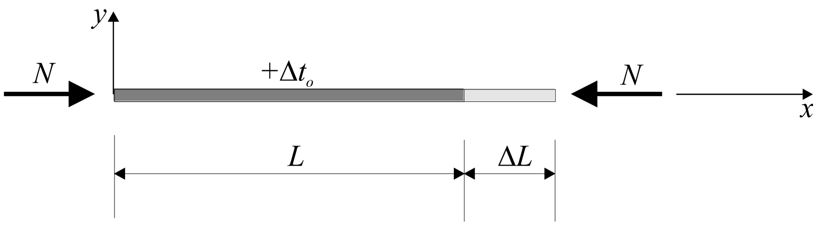


Figure 17. The element extension caused by temperature.

As we know, the increase in temperature of an element causes it to lengthen which, with the assumption of a steady increase in the temperature of the whole bar, is described by the equation:

$$\varepsilon_t = \frac{\Delta L}{L} = \alpha_t \Delta t_o\,, \tag{129}$$

where α_t is the coefficient of thermal expansion of the material from which the element is made, Δt_o stands for an increment of temperature in the middle fibres (joining centres of gravity of cross sections of an element).

We assume a steady increase in temperature in the whole section and homogeneity of the material. If we accept that the element has no freedom to grow but is limited by fixed nodes, we obtain an axial force which is set up within the element:

$$N = -\int_{\mathcal{A}} \sigma_t d\mathcal{A} = -\int_{\mathcal{A}} E\varepsilon_t d\mathcal{A} = -\int_{\mathcal{A}} E\alpha_t \Delta t_o d\mathcal{A} = -E\alpha_t \Delta t_o A\,, \tag{130}$$

where E is Young's modulus of the material and A signifies the surface area of the cross section of the element.

The nodal forces vector of the element due to the temperature, written in the local coordinate system xy, is equal to:

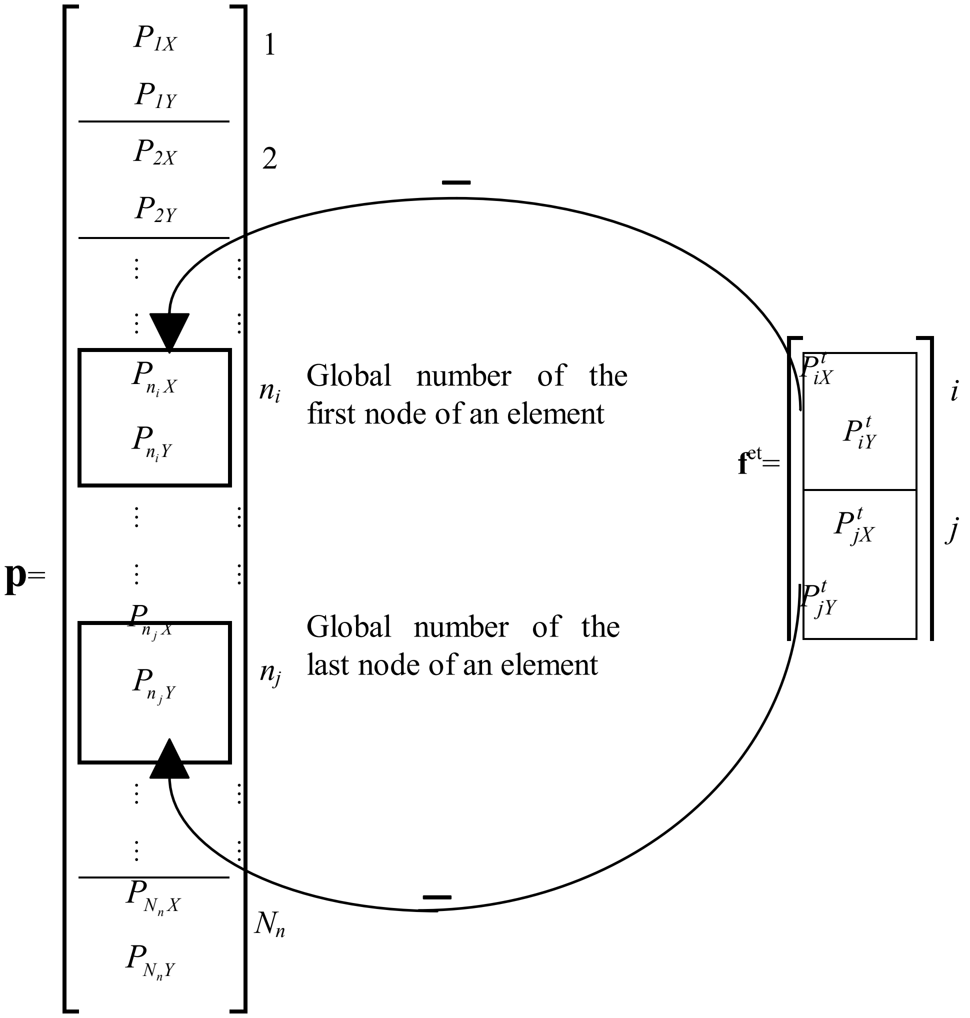
$$\mathbf{f}'^{et} = EA\alpha_t \Delta t_o \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \tag{131}$$

after transformation to the global system, with the help of relation Eqn. (92)we obtain

$$\mathbf{f}^{et} = EA\alpha_t \Delta t_o \begin{bmatrix} c \\ s \\ -c \\ -s \end{bmatrix}, \tag{132}$$

where, $c = \cos\alpha$, $s = \sin\alpha$, α - is the angle determining a slope of the loaded element with respect to the global coordinate system.

Since forces acting on the nodes are necessary for the equilibrium equations, and as it is known, they are of opposite direction to other forces acting on elements, then we subtract them from other forces while building the global nodal forces vector. This is shown inFigure 18.



$$P^t_{iX} = EA\alpha_t \Delta t_o \cos\alpha \qquad P^t_{jX} = -EA\alpha_t \Delta t_o \cos\alpha$$

$$P^t_{iY} = EA\alpha_t \Delta t_o \sin\alpha \qquad P^t_{jY} = -EA\alpha_t \Delta t_o \sin\alpha$$

Figure 18. The temperature load included into the nodal load vector.

1.13. The geometric load on a truss

The final type of truss load, which we will describe, is the geometric load (forced displacements of nodes).

We assume that the node *r* is displaced by the vector **d** (Figure 19). It is necessary to apply forces to the node to cause this displacement. Values of these forces are not known, whereas we know components of the displacement of the node *r*:

$$u_{rX} = d_X\,,\; u_{rY} = d_Y\,, \tag{133}$$

where d_X, d_Y are the components of the vector of the forced displacement **d**.

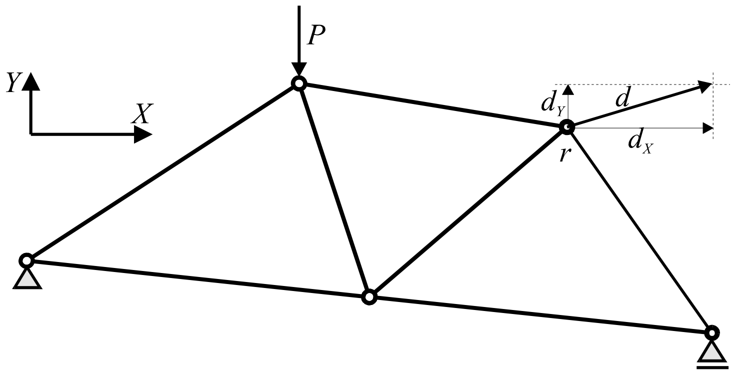


Figure 19. The scheme of the geometric load acting on the truss.

Eqn. (133)is like the known equations of the boundary conditions (108)and (109)but with one difference, here we have obtained nonhomogeneous equations. It changes the procedure of symmetrisation of the stiffness matrix. Previously we inserted zeros into suitable columns of the matrix **K** which did not induce any consequences because this matrix was multiplied by zero values of displacements of the support nodes. At this time we have to keep the components of the matrix occurring in this column because they are multiplied by given displacements (comp. Eqn. (133)) and they are usually not equal to zero.

Hence transformations of the stiffness matrix **K** and nodal loads vector **p** leading to the consideration of the geometric load should look as follows:

We form vectors $\mathbf{k}_{rX}$ and $\mathbf{k}_{rY}$ which are suitable columns of the matrix **K** joined with the displacements of the node *r*. $\mathbf{k}_{rX}$ is the column with a number equal to the displacement global number u_{rX} and $\mathbf{k}_{rY}$ is the column with a number equal to the displacement global number u_{rY}.

We move the nodal forces due to the known displacements d_X and d_Y to the right hand side of the set of equations:

$$\mathbf{p}^d = \mathbf{p} - \mathbf{k}_{rX} d_X - \mathbf{k}_{rY} d_Y \; . \tag{134}$$

We consider boundary conditions in the standard way as in Sec.2.6. However, there is one difference, we put known values into the rows of the right hand side vector $\mathbf{p}^d$. These rows have the global numbers equivalent to the degrees of freedom u_{rX} and u_{rY}.

After making the above transformations, the following set of equations rises:

$$\mathbf{K}^r \, \mathbf{u} = \mathbf{p}^{rd} \; , \tag{135}$$

where $\mathbf{K}^r$ is the stiffness matrix which is modified by the standard consideration of the boundary conditions as in Eqn. (112) and $\mathbf{p}^{rd}$ is the modified vector $\mathbf{p}^d$ determined by Eqn. (134)after inserting known values of displacements:

$$P_{rX} = d_X \; , \; P_{rY} = d_Y \; .$$

$$\mathbf{p}^d = \left[\begin{array}{c} P_{1X} \\ P_{1Y} \\ \hline P_{2X} \\ P_{2Y} \\ \hline : \\ : \\ \hline P_{rX} \\ P_{rY} \\ \hline : \\ : \\ \hline : \\ : \\ \hline : \\ : \\ \hline P_{NnX} \\ P_{NnY} \end{array}\right] \begin{array}{l} 1 \\ \\ 2 \\ \\ : \\ : \\ n_r \\ \\ : \\ : \\ : \\ : \\ : \\ : \\ N_n \\ \\ \end{array} - \mathbf{k}_{rX} \cdot d_X - \mathbf{k}_{rY} \cdot d_Y \qquad \mathbf{p}^{rd} = \left[\begin{array}{c} \; \\ \; \\ \hline \; \\ \; \\ \hline \; \\ \; \\ \hline d_X \\ d_Y \\ \hline \; \\ \; \\ \hline \; \\ \; \\ \hline \; \\ \; \\ \hline \; \\ \; \end{array}\right] \begin{array}{l} 1 \\ \\ 2 \\ \\ : \\ : \\ n_r \\ \\ : \\ : \\ : \\ : \\ : \\ : \\ N_n \\ \\ \end{array}$$

Figure 20a. Preparing the geometric load vector.

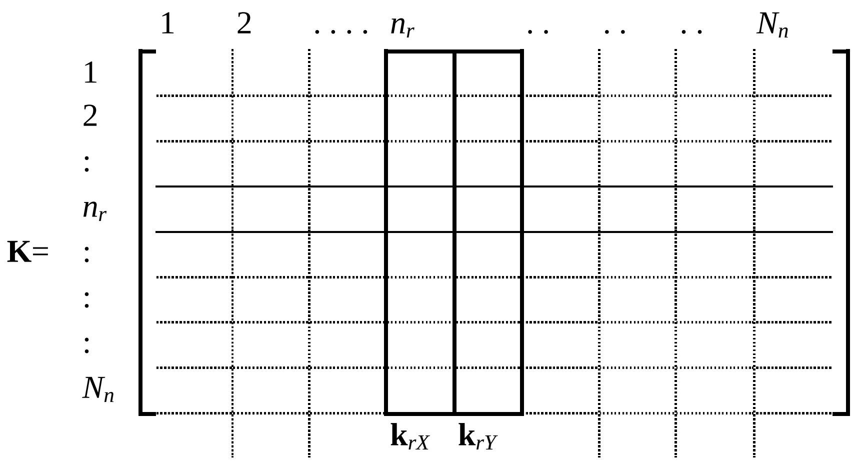


Figure 21b. The geometric load included into the nodal load vector.

1.14. Support reactions, internal forces and stresses in elements

After aggregation of the stiffness matrix, consideration of the boundary conditions and building the nodal forces vector, we obtain the set of linear equations in forms Eqn. (113),(126) or (135)with a positively determined symmetric matrix. Methods of solving such equations are described in Appendix 2. The solution of the set of equations is the nodal displacements vector of a structure. Knowing nodal displacements allows us to determine control sums of nodes and support reactions in the support nodes in a very simple way. And then we make use of Eqn. (104) in which the matrix **K** does not contain any information about the support constraints.

$$\mathbf{r} = \mathbf{K}\,\mathbf{u} - \mathbf{p} \tag{136}$$

The vector of reactions **r** should contain zeros at free nodes and values of reactions at support nodes. If we assume the occurrence of the local coordinate system in some nodes (the ‘skew’ supports), then the components of reactions will be expressed in the local coordinate system.

Since numerical errors resulting from approaching values of numbers stored in the computer memory increase during the solution process, the control sums are rarely equal to zero and they are most often small numbers, for example the order of $1{\cdot}10^{-10}$.

Components of the global displacements vector enable the building of global displacements vectors for the elements (Figure 22).

Since the components of the vector **u** are not always written in the global coordinate system (the ‘skew’ supports), then it can happen that some components of the vector $\mathbf{u}^e$ are expressed in the global system and others are expressed in the local coordinate system. To simplify further discussion we standardise the description of the vector bringing down the components to the global coordinate system by taking advantage of Eqn. (118). It should be noted that it is only necessary for elements joined to a node which is supported by a skew support.

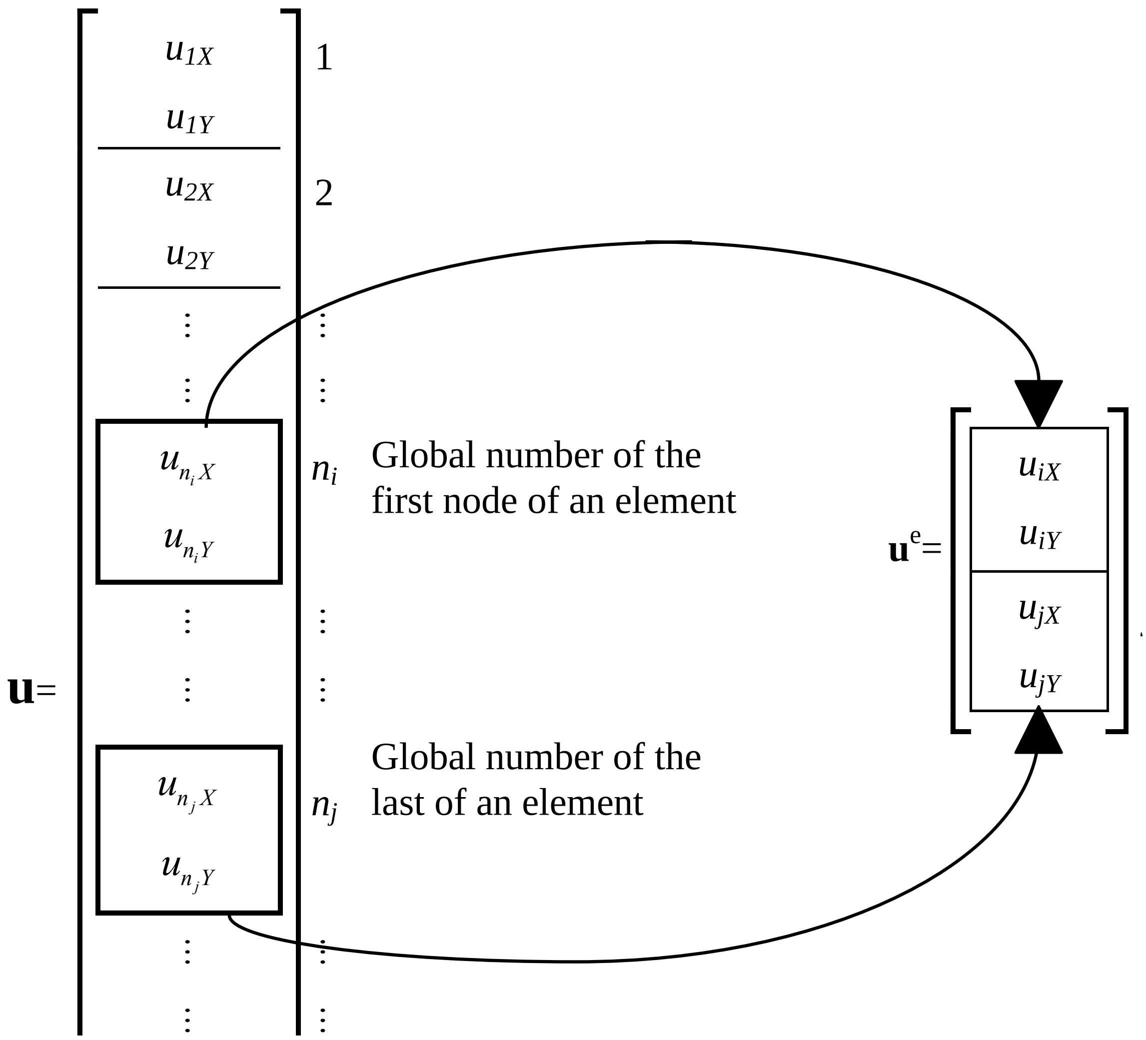


Figure 22. The geometric load included into the global stiffness matrix.

Nodal displacements of an element allow the internal force N in a truss element to be calculated quite easily. We can either make use of Eqn. (72) which requires knowledge of displacements in the local coordinate system of the element or on the basis of Eqn. (70), (74)and (93)we search the relationship:

$$N = \frac{EA}{L}\left[c\left(u_{jX} - u_{iX}\right) + s\left(u_{jY} - u_{iY}\right)\right], \tag{137}$$

where similarly to Eqn. (79) $c = \cos\alpha$ and $s = \sin\alpha$.

Stresses in the truss element, assuming that the bar is homogeneous, are the axial stresses only which can be calculated using a simple relationship:

$$\sigma_x = \frac{N}{A} = \frac{E}{L}\left[c\left(u_{jX} - u_{iX}\right) + s\left(u_{jY} - u_{iY}\right)\right]. \tag{138}$$

If the element is loaded with a temperature gradient, then the correction coming from thermal expansion of the material shown in Eqn. (137) and (138)should be taken into consideration:

$$\sigma_x = E\left(\varepsilon - \varepsilon_t\right) = \frac{E}{L}\left[c\left(u_{jX} - u_{iX}\right) + s\left(u_{jY} - u_{iY}\right) - L\left(\alpha_t \Delta t_o\right)\right] \tag{139}$$

and

$$N = A\sigma_x = \frac{EA}{L}\left[c\left(u_{jX} - u_{iX}\right) + s\left(u_{jY} - u_{iY}\right) - L\left(\alpha_t \Delta t_o\right)\right]. \tag{140}$$

The calculation of displacements, constrained reactions and internal forces in the element completes the static analysis of the truss.

2. 3D truss structures

Although 3D truss structures have been around for a long time (comp. Timoshenko and Goodier (1962)), they have been used very rarely until now. They are particularly difficult to solve. Though a series method simplifying the calculation of internal forces (the method of nodal equilibrium and its graphic variant - Cremona's method and the method of sections - Ritter's method, etc.) has been devised for statically determined plane trusses, in case of space trusses, only the method of nodal equilibrium has remained. Large sets of equations which are generated by this method for space trusses have discouraged engineers from designing this type of structure. 3D structures looking like trusses, in fact, are seldom trusses. For instance, the famous Eiffel's tower or support columns of overhead power lines, masts (in particular with the quadrangular crosses) are most often space frames because they keep their geometric stability thanks to bent elements which do not exist in classical trusses. Both the use of computers and new methods of statics analysis of a structure making use of new technical possibilities (the finite element method is one of the main methods among them) have enabled considerable progress in designing space trusses.

One of the most popular uses of these structures is in structural roofs. Examples of space trusses are presented in Figure 23.

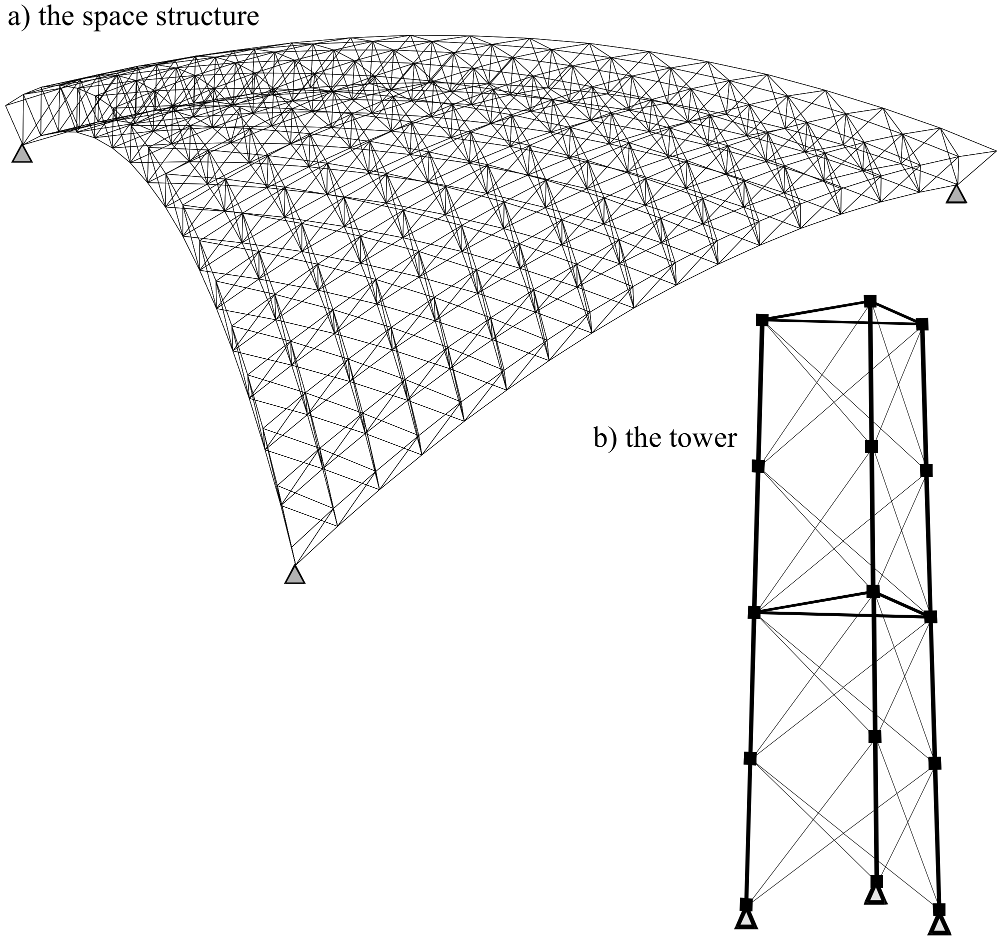


Figure 23. The example of 3D trusses.

2.1. Notation and basic relations

The node of a space truss has three degrees of freedom because in order to describe its movement, we have to give three components of a displacement vector. The displacement vector and forces acting on an element of the space truss are shown inFigure 24. As in Chapter 2 components of forces and displacements vector are collected in column matrices which will be called vectors;

- nodal displacements vector of the first node i in the global coordinate system:

$$\mathbf{u}_i = \begin{bmatrix} u_{iX} \\ u_{iY} \\ u_{iZ} \end{bmatrix}, \tag{141}$$

– the same vector in the local coordinate system:

$$\mathbf{u'}_i = \begin{bmatrix} u_{ix} \\ u_{iy} \\ u_{iz} \end{bmatrix}, \tag{142}$$

– vector of nodal forces acting at the first node i of an element written in the global system:

$$\mathbf{f}_i = \begin{bmatrix} F_{iX} \\ F_{iY} \\ F_{iZ} \end{bmatrix}, \tag{143}$$

and in the local system:

$$\mathbf{f'}_i = \begin{bmatrix} F_{ix} \\ F_{iy} \\ F_{iz} \end{bmatrix}. \tag{144}$$

The above vectors form forces and displacements vectors of an element:

– vector of the nodal displacements of an element e with the node i (the first one) and j (the last one) is written in the global coordinate system as follows:

$$\mathbf{u}^e = \begin{bmatrix} \mathbf{u}_i \\ \mathbf{u}_j \end{bmatrix} = \begin{bmatrix} u_{iX} \\ u_{iY} \\ u_{iZ} \\ u_{jX} \\ u_{jY} \\ u_{jZ} \end{bmatrix}, \tag{145}$$

– its description in the local system is:

$$\mathbf{u'}^e = \begin{bmatrix} \mathbf{u'}_i \\ \mathbf{u'}_j \end{bmatrix} = \begin{bmatrix} u_{ix} \\ u_{iy} \\ u_{iz} \\ u_{jx} \\ u_{jy} \\ u_{jz} \end{bmatrix}. \tag{146}$$

– vector of the nodal forces of an element in the global system

$$\mathbf{f}^{e} = \begin{bmatrix} \mathbf{f}_{i} \\ \mathbf{f}_{j} \end{bmatrix} = \begin{bmatrix} F_{iX} \\ F_{iY} \\ F_{iZ} \\ F_{jX} \\ F_{jY} \\ F_{jZ} \end{bmatrix}, \tag{147}$$

and in the local system

$$\mathbf{f'}^{e} = \begin{bmatrix} \mathbf{f'}_{i} \\ \mathbf{f'}_{j} \end{bmatrix} = \begin{bmatrix} F_{ix} \\ F_{iy} \\ F_{iz} \\ F_{jx} \\ F_{jy} \\ F_{jz} \end{bmatrix}. \tag{148}$$

Interpretation and meanings of the symbols used here can be found inFigure 24.

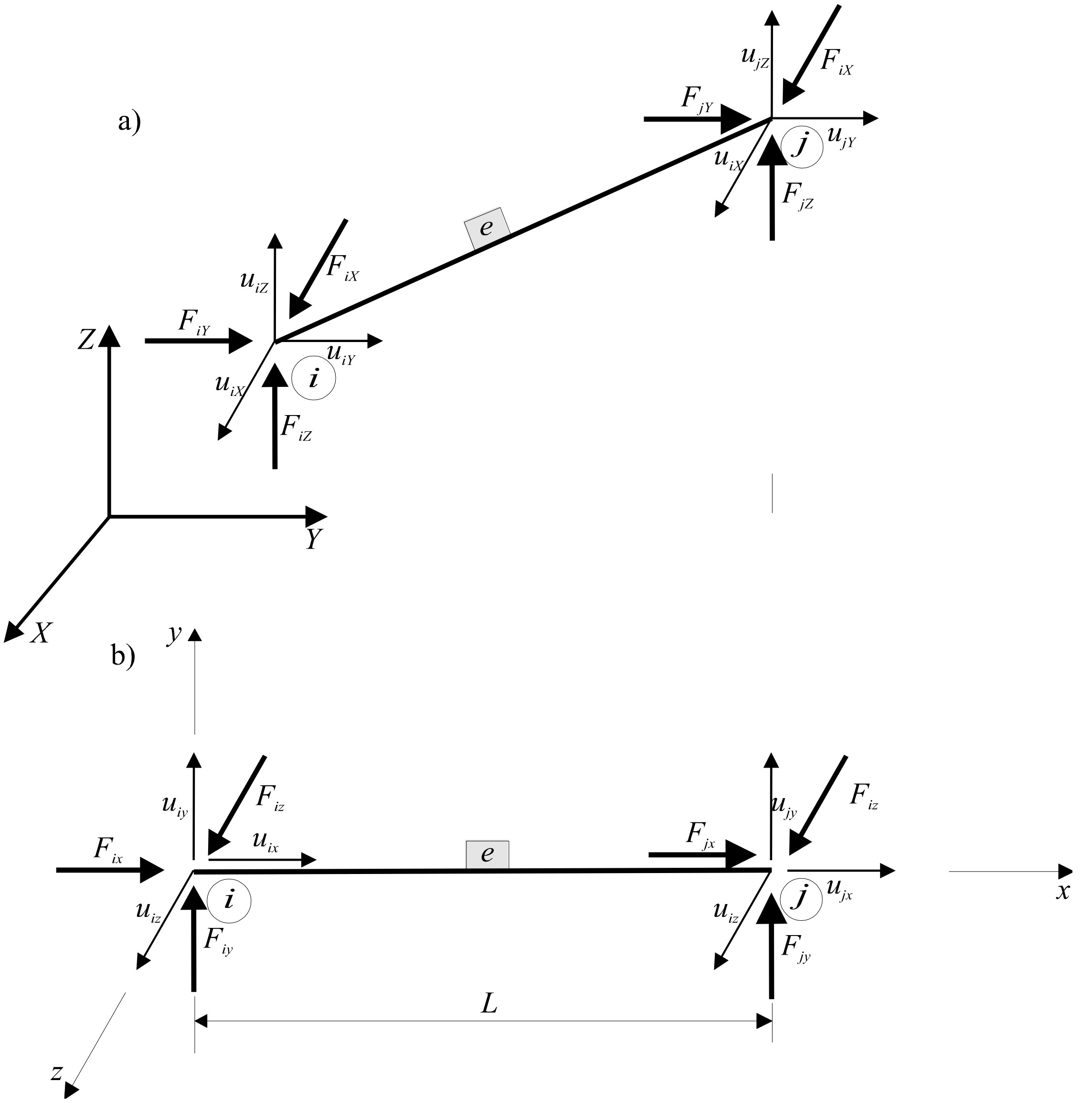


Figure 24. Nodal forces and displacements for the 3D truss element: a) in the global coordinate system; b) in the element local coordinate system.

2.2. The element stiffness matrix of a space truss

The relationship between nodal forces and nodal displacements for a space truss is identical to that for a plane truss if we analyse it in the local coordinate system. Obviously, the third force is F_{iz} or F_{jz} but the equilibrium equation of moments with respect to the y axis results in the zero value of this force:

$$\sum F_x = F_{ix} + F_{jx} = 0 \rightarrow F_{ix} = -F_{jx} ,$$

$$\sum F_y = F_{iy} + F_{jy} = 0 \xrightarrow{\text{after considering eq. f}} F_{iy} = 0 , \tag{149}$$

$$\sum F_z = F_{iz} + F_{jz} = 0 \xrightarrow{\text{after considering eq. e}} F_{iz} = 0,$$

$$\sum M_x = 0,$$

$$\sum M_y = -F_{jz} L = 0 \rightarrow F_{jz} = 0,$$

$$\sum M_z = -F_{jy} L = 0 \rightarrow F_{jy} = 0.$$

The relationship between an axial force and displacements which is identical to the relation presented in Chapter 2 (comp. Eqn. (72)) allows us to express the searched dependence as follows:

$$\mathbf{f}'^e = \mathbf{K}'^e \mathbf{u}'^e, \tag{150}$$

where

$$\mathbf{K}'^e = \begin{bmatrix} \mathbf{J}' & -\mathbf{J}' \\ -\mathbf{J}' & \mathbf{J}' \end{bmatrix}, \tag{151a}$$

$$\mathbf{J}' = \frac{EA}{L} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{151b}$$

The transformation of these equations from the local system to the global one will be done analogously to the transformation performed in case of a 2D truss (Eqn. (91), (95), (96)).

In order to complete the transformation of the element stiffness matrix to the global system, we need the rotation matrix of a node $\mathbf{R}_i$, and then we can determine components of the matrix $\mathbf{J}$ similar to those described by Eqn.(98).

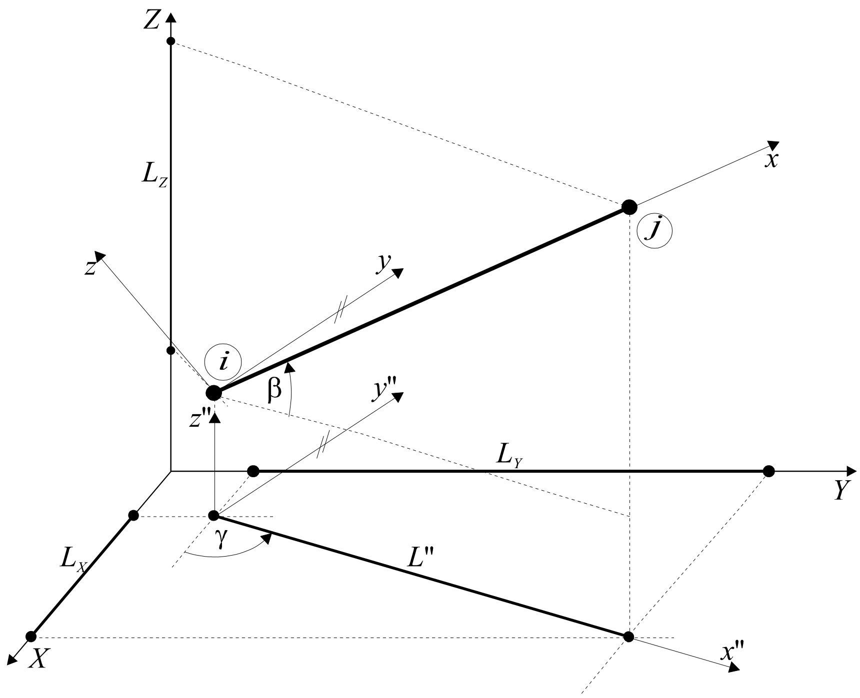


Figure 25. The truss element arrangement with regard to the global coordinate system.

Since the location of the *y* and *z* axes of the local system is not essential for truss elements, we will choose the direction of the *y* axis in such a way that it will always be parallel to the *XY* plane of the global system but for bars parallel to the *Z* axis there will be an additional assumption that the *y* axis is parallel to the *Y* axis (comp.Figure 25).

The rotation from the local coordinate system to the global one will be composed of two intermediate rotations. First, we rotate the system *xyz* to the intermediate system *x*"*y*"*z*" selected so that the *x*" axis is parallel to the *XY* plane and next we rotate the system *x*"*y*"*z*" by an angle γ so that the *x*" and *X* axes are parallel. The first rotation around the *y* axis gives the following result:

$$\begin{bmatrix} u_{x"} \\ u_{y"} \\ u_{z"} \end{bmatrix} = \begin{bmatrix} c_\beta & 0 & -s_\beta \\ 0 & 1 & 0 \\ s_\beta & 0 & c_\beta \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix},$$

or in a shorter form $\mathbf{u}'' = \mathbf{R}_\beta \mathbf{u}'$, (152)

where $c_\beta = \cos\beta = \dfrac{L''}{L}$, $s_\beta = \sin\beta = \dfrac{L_Z}{L}$, $L_X = X_j - X_i$, $L_Y = Y_j - Y_i$, $L_Z = Z_j - Z_i$,

$L'' = \sqrt{L_X^2 + L_Y^2}$, $L = \sqrt{L''^2 + L_Z^2}$.

The second rotation around the z axis leads the equations to the global system:

$$\begin{bmatrix} u_X \\ u_Y \\ u_Z \end{bmatrix} = \begin{bmatrix} c_\gamma & -s_\gamma & 0 \\ s_\gamma & c_\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{x''} \\ u_{y''} \\ u_{z''} \end{bmatrix}$$

or in a shorter form $\mathbf{u} = \mathbf{R}_\gamma \mathbf{u}''$, (153)

where $c_\gamma = \cos\gamma = \dfrac{L_X}{L''}$, $s_\gamma = \sin\gamma = \dfrac{L_Y}{L''}$,

when L"=0 we assume γ=0, hence $c_\gamma = 1$ and $s_\gamma = 0$.

The composition of both rotations which means putting Eqn. (151) into(152), gives the searched rotation matrix of a node

$$\mathbf{u}_i = \mathbf{R}_{i\gamma}\mathbf{R}_{i\beta}\mathbf{u}'_i \,, \tag{154}$$

where $\mathbf{R}_i = \mathbf{R}_{i\gamma}\mathbf{R}_{i\beta}$.

After multiplying matrices $\mathbf{R}_{i\gamma}\mathbf{R}_{i\beta}$, we obtain the final form of the rotation matrix $\mathbf{R}_i$:

$$\mathbf{R}_i = \begin{bmatrix} c_\gamma c_\beta & -s_\gamma & -c_\gamma s_\beta \\ s_\gamma c_\beta & c_\gamma & -s_\gamma s_\beta \\ s_\beta & 0 & 0 \end{bmatrix}. \tag{155}$$

We calculate the transformation of the block **J** of the element stiffness matrix of the space truss from the local coordinate system to the global one as in Chapter 2 (comp. the similar transformation of the stiffness matrix Eqn. (95))

$$\mathbf{J} = \mathbf{R}_i \mathbf{J}' \left(\mathbf{R}_i\right)^{\mathrm{T}} . \tag{156}$$

Inserting relations Eqn. (150b) and (154) into the above equation we obtain:

$$\mathbf{J} = \frac{EA}{L} \begin{bmatrix} \left(c_\gamma c_\beta\right)^2 & c_\gamma s_\gamma \left(c_\beta\right)^2 & c_\gamma c_\beta s_\beta \\ c_\gamma s_\gamma \left(c_\beta\right)^2 & \left(s_\gamma c_\beta\right)^2 & s_\gamma c_\beta c_\beta \\ c_\gamma c_\gamma s_\beta & s_\gamma c_\beta s_\beta & \left(s_\beta\right)^2 \end{bmatrix}. \tag{157}$$

After the introduction of a convenient notation:

$$C_X = \frac{L_X}{L}, \quad C_Y = \frac{L_Y}{L}, \quad C_Z = \frac{L_Z}{L} \tag{158}$$

which are called direction cosines of an element, we obtain a very simple form of the block $\mathbf{J}$ of the stiffness matrix:

$$\mathbf{J} = \frac{EA}{L}\begin{bmatrix} C_X^2 & C_X C_Y & C_X C_Z \\ C_X C_Y & C_Y^2 & C_Y C_Z \\ C_X C_Z & C_Y C_Z & C_Z^2 \end{bmatrix}. \tag{159}$$

Relation Eqn. (158) obtained after being inserted into Eqn. (151a) gives us the element stiffness matrix for the space truss in the global coordinate system.

2.3. The vector of temperature loads for an element of 3D truss

Since forming a loads vector of a truss for concentrated forces is identical to forming it for a 2D truss, we will also not discuss the vector $\mathbf{p}$. On the other hand, we will discuss the vector of nodal forces due to a temperature load. Components of this vector in the local coordinate system are identical (apart from the correction in reference to the third component of the vector!) to the components of the vector for a plane truss Eqn.(131).

$$\mathbf{f}'^{et} = EA\alpha_t \Delta t_o \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}. \tag{160}$$

The transformation to the global system proceeds in agreement with Eqn. (92) in the following way:

$$\mathbf{f}^{et} = \mathbf{R}^e \, \mathbf{f}'^{et}, \tag{161}$$

where $\mathbf{R}^e$ is the element rotation matrix:

$$\mathbf{R}^e = \begin{bmatrix} \mathbf{R}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_j \end{bmatrix}. \tag{162}$$

Since a truss element is straight, $\mathbf{R}_i = \mathbf{R}_j$, where the matrix $\mathbf{R}_i$ is defined by Eqn. (154).

After inserting Eqn. (154) into (160) and multiplying them, we obtain

$$
\mathbf{f}^{et} = EA\alpha_t \Delta t_o \begin{bmatrix} c_\gamma c_\beta \\ s_\gamma c_\beta \\ s_\beta \\ -c_\gamma c_\beta \\ -s_\gamma c_\beta \\ -s_\beta \end{bmatrix} \tag{163}
$$

or in another form:

$$
\mathbf{f}^{et} = EA\alpha_t \Delta t_o \begin{bmatrix} C_X \\ C_Y \\ C_Z \\ -C_X \\ -C_Y \\ -C_Z \end{bmatrix}. \tag{164}
$$

The remaining procedure is identical to the one employed in case of a plane truss.

2.4. The boundary element

In Chapter 2, we explained widely different types of boundary conditions and also elastic boundary elements. Since they are very useful elements for modelling many different boundary conditions, we will pay more attention to them in this chapter concentrating on differences between plane and space trusses.

We will discuss the most general elastic element with stiffness k_b dropping with respect to axes of the global system with the angles α_X, α_Y, α_Z whose direction cosines are equal to

$$
c_X = \cos\alpha_X,\ c_Y = \cos\alpha_Y,\ c_Z = \cos\alpha_Z. \tag{165}
$$

The stiffness matrix of this element in the local system is analogous to the matrix stiffness of an ordinary truss element but this element has three degrees of freedom, so the stiffness matrix contains only one block $\mathbf{J}'$ (Eqn. (151b)

$$
\mathbf{K}'^{b} = k_b \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{166}
$$

Transforming this element to the global coordinate system we obtain a matrix which is very similar to the one obtained in Chapter 2 for a plane truss:

$$\mathbf{K}^b = k_b \begin{bmatrix} c_X^2 & c_X c_Y & c_X c_Z \\ c_X c_Y & c_Y^2 & c_Y c_Z \\ c_X c_Z & c_Y c_Z & c_Z^2 \end{bmatrix} \tag{167}$$

Boundary elements can form for example, an element with three different types of stiffness k_x, k_y, k_z parallel to axes of the local system xyz:

$$\mathbf{K}'^b = \begin{bmatrix} k_x & 0 & 0 \\ 0 & k_y & 0 \\ 0 & 0 & k_z \end{bmatrix}. \tag{168}$$

The transformation of this matrix to the global system is analogous to the transformation of the block $\mathbf{J}'$ Eqn.(156) discussed earlier. We do not give the result of this transformation here leaving its execution as an exercise for the reader.

2.5. Stresses and Internal forces

As in Sec.2.11 of Chapter 2, we present here equations to calculate stresses and internal forces in an element:

$$\sigma_x = E(\varepsilon - \varepsilon_t) = \frac{E}{L}\left[\left(u_{jx} - u_{ix}\right) - L(\alpha_t \Delta t_o)\right], \tag{169}$$

or in another form:

$$\sigma_x = \frac{E}{L}\begin{bmatrix}-1 & 0 & 0 & 1 & 0 & 0\end{bmatrix}\mathbf{u}'^e - E\alpha_t \Delta t_o. \tag{170}$$

The transformation of the vector $\mathbf{u}'^e$ to the global system gives the relationship:

$$\sigma_x = \frac{E}{L}\begin{bmatrix}-1 & 0 & 0 & 1 & 0 & 0\end{bmatrix}\left(\mathbf{R}^e\right)^{\mathsf{T}} \mathbf{u}^e - E\alpha_t \Delta t_o \tag{171}$$

which, after multiplication, gives components of direct stress in an element as follows:

$$\sigma_x = E\left\{\begin{bmatrix}-\mathbf{c}^{\mathsf{T}} & \mathbf{c}^{\mathsf{T}}\end{bmatrix}\left(\mathbf{R}^e\right)^{\mathsf{T}} \mathbf{u}^e \frac{1}{L} - \alpha_t \Delta t_o\right\} \tag{172}$$

where $\mathbf{c}$ is the vector of element direction cosines: $\mathbf{c}^{\mathsf{T}} = \begin{bmatrix} c_X & c_Y & c_Z \end{bmatrix}$ (158).

Calculating the normal force consists of integrating stresses on the surface of a cross section with an assumption of homogeneity of the stress field (as in Chapter 2)

$$N = \sigma_x A = EA\left\{\frac{1}{L}\begin{bmatrix}-\mathbf{c}^{\mathsf{T}} & \mathbf{c}^{\mathsf{T}}\end{bmatrix}\left(\mathbf{R}^e\right)^{\mathsf{T}} \mathbf{u}^e - \alpha_t \Delta t_o\right\}. \tag{173}$$

The remaining support reactions are calculated with the help of Eqn. (136). We do it exactly in the same way as it has been done for the 2D truss, so we will not describe here the above problem for a space truss in detail.

3. 2D frame systems

The correct choice of model for a structure is very important for quality and exactness of the results obtained. The choice of frame or truss (for example, a truss with fixed nodes) is often subjective and it depends on experience and intuition of the analyst.

In this chapter, we will present the following model of a bar structure - a 2D frame which gives more possibilities of modelling real structures. The element of a 2D frame is more general than a truss element presented in Chapter 2 because with help of this element we can also model ideal truss structures (articulated connection of elements at nodes). We can simply say that a frame is a structure whose bars can be bent while truss elements can be only compressed and stretched. It has the following consequences:

- bar (an element) of a frame can be loaded between nodes,
- modelling of different types of loads is possible, for example: concentrated forces, concentrated moments, distributed loads,
- connection of an element with a node can be a fixed connection provided that the rotation of a node and of a nodal section of the element are identical or it can be an articulated connection when independent rotations of a node and a nodal section are possible,
- node of a 2D frame has three degrees of freedom which means that we have to know two components of a translation vector: u_X, u_Y and the rotation angle φ_Z in order to determine the location of this node.

In the case of plane frames, we will neglect index Z of rotation angles in our notation because all rotation angles on the plane XY (which we will use to describe the structure) are rotations with respect to the Z axis. Let us assume that a frame element is straight and homogeneous which means that it is made from a homogeneous material and has a constant cross section. The view of a frame element, directions, senses of nodal displacements and forces which we will consider as positive are presented in Figure 26.

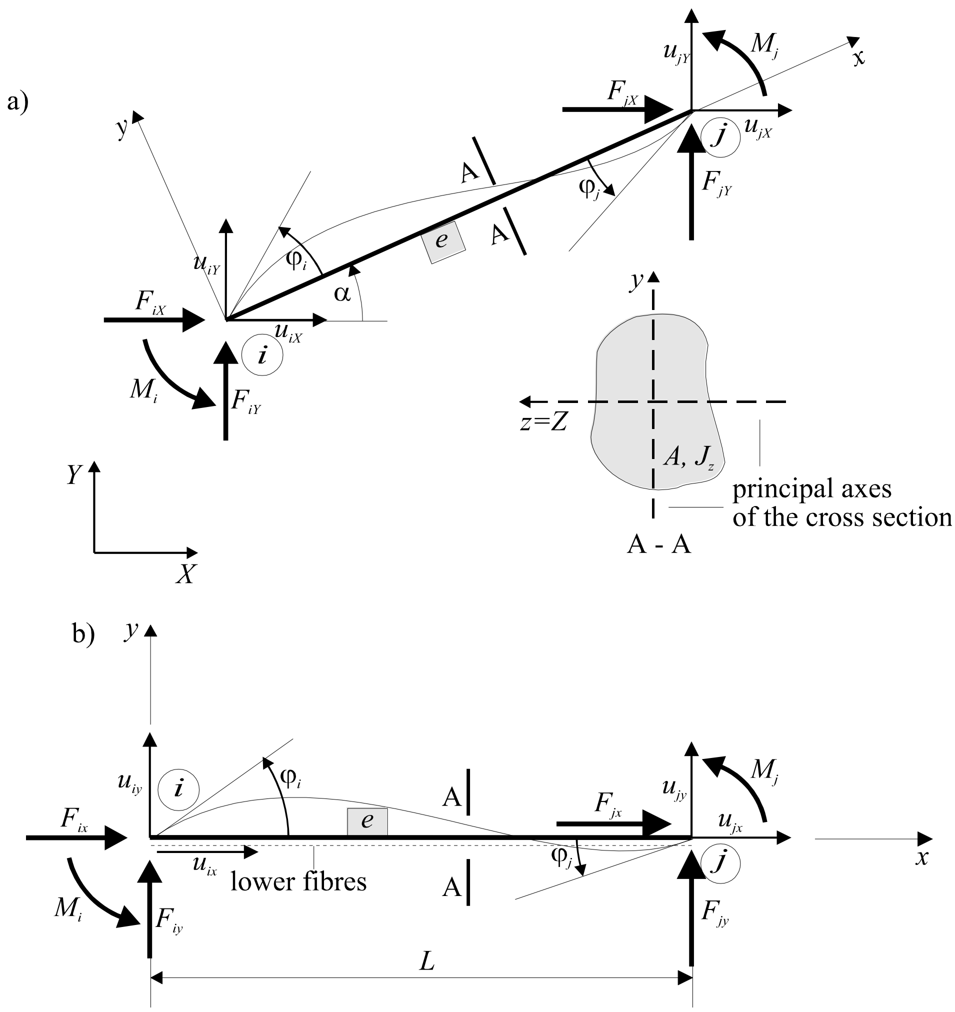


Figure 26Nodal loads and displacements for the plane frame element: a) in the global coordinate system; b) in the element local coordinate system.

3.1. The element stiffness matrix for a 2D frame

We group nodal displacements and forces shown in Figure 26a,b in column matrices just as we did previously in Chapters 2 and 3. They are called vectors:

– displacement vector of the first node i and the last node j in the local system (Figure 26b)

$$\mathbf{u}'_i = \begin{bmatrix} u_{ix} \\ u_{iy} \\ \varphi_i \end{bmatrix}, \quad \mathbf{u}'_j = \begin{bmatrix} u_{jx} \\ u_{jy} \\ \varphi_j \end{bmatrix}. \tag{174}$$

– nodal forces vector in the local coordinate system

$$\mathbf{f'}_i = \begin{bmatrix} F_{ix} \\ F_{iy} \\ M_i \end{bmatrix}, \quad \mathbf{f'}_j = \begin{bmatrix} F_{jx} \\ F_{jy} \\ M_j \end{bmatrix}. \tag{175}$$

– element displacement vector in the local coordinate system

$$\mathbf{u'}^e = \begin{bmatrix} \mathbf{u'}_i \\ \mathbf{u'}_j \end{bmatrix} = \begin{bmatrix} u_{ix} \\ u_{iy} \\ \varphi_i \\ u_{jx} \\ u_{jy} \\ \varphi_j \end{bmatrix}. \tag{176}$$

– element forces vector in the local coordinate system

$$\mathbf{f'}^e = \begin{bmatrix} \mathbf{f'}_i \\ \mathbf{f'}_j \end{bmatrix} = \begin{bmatrix} F_{ix} \\ F_{iy} \\ M_i \\ F_{jx} \\ F_{jy} \\ M_j \end{bmatrix}. \tag{177}$$

We can also describe all the vectors formulated above in the global system:

$$\mathbf{u}_i = \begin{bmatrix} u_{iX} \\ u_{iY} \\ \varphi_i \end{bmatrix}, \ \mathbf{u}_j = \begin{bmatrix} u_{jX} \\ u_{jY} \\ \varphi_j \end{bmatrix}, \tag{178}$$

$$\mathbf{f}_i = \begin{bmatrix} F_{iX} \\ F_{iY} \\ M_i \end{bmatrix}, \quad \mathbf{f}_j = \begin{bmatrix} F_{jX} \\ F_{jY} \\ M_j \end{bmatrix}, \tag{179}$$

$$\mathbf{u}^e = \begin{bmatrix} \mathbf{u}_i \\ \mathbf{u}_j \end{bmatrix} = \begin{bmatrix} u_{iX} \\ u_{iY} \\ \varphi_i \\ u_{jX} \\ u_{jY} \\ \varphi_j \end{bmatrix}, \tag{180}$$

$$\mathbf{f}^e = \begin{bmatrix} \mathbf{f}_i \\ \mathbf{f}_j \end{bmatrix} = \begin{bmatrix} F_{iX} \\ F_{iY} \\ M_i \\ F_{jX} \\ F_{jY} \\ M_j \end{bmatrix}. \tag{181}$$

As in the previous chapters, the relationship between nodal forces and nodal displacements will be of great importance. This relation (analogous to Eqn. (66) for a truss) in the local coordinate system has the form:

$$\mathbf{K}'^e \, \mathbf{u}'^e = \mathbf{f}'^e, \tag{182}$$

and in the global system

$$\mathbf{K}^e \, \mathbf{u}^e = \mathbf{f}^e. \tag{183}$$

At the moment, we will concentrate on searching for the stiffness matrix $\mathbf{K}'^e$ in the local coordinate system and next its transformation to the global system.

Equilibrium equations of the element presented in Figure 26b lead to the following relations between nodal forces:

$$\sum F_x = F_{ix} + F_{jx} = 0 \rightarrow F_{ix} = -F_{jx} \; ;$$

$$\sum F_y = F_{iy} + F_{jy} = 0 \; ; \tag{184}$$

$$\sum M_i = M_i + M_j + F_{jy} L = 0 \, .$$

It has been shown that three equations are unable to calculate six components of the vector $\mathbf{f}'^e$. The discussion concerning element strains will provide these missing equations. The deformation caused by the axial forces F_{ix} and F_{jx} is identical to the deformation of a truss element, hence we take advantage of previously determined dependence Eqn. (72) and Eqn. (73a). We will obtain the remaining equations when we consider the flexural deformation of an element and the relationship between shearing forces and bending moments. The well-known relationship between curvature and bending moment is (comp. Jastrzębski et al. (1985)):

$$\frac{1}{\rho} = \frac{\dfrac{d^2y}{dx^2}}{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}} = \frac{M(x)}{EJ_z}, \tag{185}$$

where ρ represents the radius of a curvature, E is Young's modulus of a material, J_z is the moment of inertia of an element cross section (comp.Figure 26a).

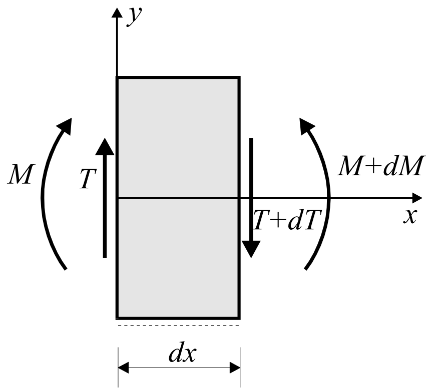


Figure 27. The segment of the bar element with internal forces.

The equilibrium of one section of a bar in bending (Figure 27) gives the equation:

$$T(x) = \frac{dM(x)}{dx}. \tag{186}$$

Since we are dealing with linear structures with small deflections, we assume $\frac{dy}{dx} << 1$, which simplifies Eqn. (185) to the well-known form:

$$\frac{d^2y}{dx^2} = \frac{M(x)}{EJ_z}. \tag{187}$$

The opposite sign of the right hand side of Eqn. (185) and (187) (comp. Jastrzębski et al. (1985)) to the one that we have usually assumed, comes from the sense of the y axis of the local coordinate system which is orientated anticlockwise in our assumptions.

Differentiating this equation twice, we obtain the relationship (comp. Jastrzębski et al. (1985), Nowacki (1976)):

$$\frac{d^4 y}{dx^4} = \frac{q_y(x)}{EJ_z}, \tag{188}$$

where $q_y(x)$ denotes the distributed load which is perpendicular to the axis of an element. Here the element is free from nodal loads, thus, $q_y \equiv 0$.

Finally, we obtain the set of differential equations:

$$\begin{aligned} &\text{a)} \quad \frac{d^4 y}{dx^4} = 0, \\ &\text{b)} \quad \frac{d^2 y}{dx^2} = \frac{M(x)}{EJ_z}, \\ &\text{c)} \quad \frac{d^3 y}{dx^3} = \frac{T(x)}{EJ_z}. \end{aligned} \tag{189}$$

After integrating relations Eqn. (189a) we obtain the following equations:

- bending line of the frame element:

$$y(x) = C_1 \frac{x^3}{6} + C_2 \frac{x^2}{2} + C_3 x + C_4, \tag{190}$$

- bending moment:

$$M(x) = EJ_z \left[C_1 x + C_2 \right], \tag{191}$$

- shearing force:

$$T(x) = EJ_z C_1, \tag{192}$$

where C_1 ... C_4 are integration constants which should be determined on the basis of boundary conditions.

We have four boundary conditions:

- at node i , $x = 0$:

$$\begin{aligned} &y(0) = u_{iy}, \\ &\left. \frac{dy}{dx} \right|_{x=0} = \varphi_i, \end{aligned} \tag{193}$$

- at node j , $x=L$:

$$y(L) = u_{jy}, \tag{194}$$

$$\left.\frac{dy}{dx}\right|_{x=L} = \varphi_j \,.$$

After inserting these conditions into Eqn. (190), we obtain the following values of the integration constants:

$$C_1 = \frac{6}{L^2}\left(\varphi_i + \varphi_j - 2\frac{u_{jy} - u_{iy}}{L}\right),$$

$$C_2 = -\frac{1}{L}\left(4\varphi_i + 2\varphi_j - 6\frac{u_{jy} - u_{iy}}{L}\right), \tag{195}$$

$$C_3 = \varphi_i \,,$$

$$C_4 = u_{iy} \,.$$

Hence after putting the above equations into Eqn. (191), (192)and considering the senses of both nodal and bending moments (comp. Figure 26 andFigure 27), we obtain

$$M_i = -M(0) = \frac{EJ_z}{L}\left[4\varphi_i + 2\varphi_j - 6\frac{u_{jy} - u_{iy}}{L}\right],$$

$$M_j = M(L) = \frac{EJ_z}{L}\left[2\varphi_i + 4\varphi_j - 6\frac{u_{jy} - u_{iy}}{L}\right],$$

$$F_{iy} = T(0) = \frac{EJ_z}{L^2}\left[6\varphi_i + 6\varphi_j - 12\frac{u_{jy} - u_{iy}}{L}\right], \tag{196}$$

$$F_{jy} = -T(L) = \frac{EJ_z}{L^2}\left[-6\varphi_i - 6\varphi_j + 12\frac{u_{jy} - u_{iy}}{L}\right].$$

Finally, tabulating Eqn. (73a) and (196)in a suitable sequence we obtain the stiffness matrix:

$$
\mathbf{K}'^{e} = \begin{bmatrix}
\frac{EA}{L} & & & -\frac{EA}{L} & & \\
& 12\frac{EJ_z}{L^3} & 6\frac{EJ_z}{L^2} & & -12\frac{EJ_z}{L^3} & 6\frac{EJ_z}{L^2} \\
& 6\frac{EJ_z}{L^2} & 4\frac{EJ_z}{L} & & -6\frac{EJ_z}{L^2} & 2\frac{EJ_z}{L} \\
-\frac{EA}{L} & & & \frac{EA}{L} & & \\
& -12\frac{EJ_z}{L^3} & -6\frac{EJ_z}{L^2} & & 12\frac{EJ_z}{L^3} & -6\frac{EJ_z}{L^2} \\
& 6\frac{EJ_z}{L^2} & 2\frac{EJ_z}{L} & & -6\frac{EJ_z}{L^2} & 4\frac{EJ_z}{L}
\end{bmatrix}, \tag{197}
$$

The relationships described by Eqn. (196) are called transformation formulae of the displacement method in structural mechanics (in some other form) (comp. Nowacki (1976)).

3.2. Transformation of the stiffness matrix from the global coordinate system to the local one

The transfer of the matrix $\mathbf{K}'^{e}$ to the global coordinate system is done according to rules analogous to the rules described by Eqn. (75) in Sec.2.4. In order to obtain the transformation matrix of an element, we need $\mathbf{R}_i$ that is, the transformation matrix from the local system to the global one for the node i. Since the third degree of freedom of frame nodes is a rotation with respect to the z axis which does not change its location because it is always perpendicular to the plane xy, the rotation will be the same as for a truss element:

$$u_{iX} = u_{ix}\cos\alpha - u_{iy}\sin\alpha,$$

$$u_{iY} = u_{ix}\sin\alpha + u_{iy}\cos\alpha,$$

$$\varphi_{iZ} = \varphi_{iz} = \varphi_i,$$

or in the matrix form

$$
\begin{bmatrix} u_{iX} \\ u_{iY} \\ \varphi_i \end{bmatrix} = \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{ix} \\ u_{iy} \\ \varphi_i \end{bmatrix}, \quad \mathbf{u}_i = \mathbf{R}_i \mathbf{u}'_i,
$$

where $\mathbf{R}_i = \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (198)

In accordance with the assumption accepted in the introduction that the frame element is straight, the transformation matrix of the node j is identical to $\mathbf{R}_i$ which leads to the final form of the element stiffness matrix:

$$\mathbf{R}^e = \begin{bmatrix} c & -s & 0 & 0 & 0 & 0 \\ s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & -s & 0 \\ 0 & 0 & 0 & s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{199}$$

After multiplying matrices described by Eqn. (95) we obtain the stiffness matrix of a frame element in the global coordinate system. Unfortunately, its form is rather complex:

$$\mathbf{K}^e = \frac{EJ_z}{L^2}\begin{array}{l} \begin{array}{cccccc} \quad u_{ix} & \qquad u_{iy} & \qquad \varphi_{ix} & \quad u_{jx} & \qquad u_{jy} & \qquad \varphi_{jx} \end{array} \\ \left[\begin{array}{cccccc} \frac{1}{L}\left(\frac{c^2}{\lambda^2}+12s^2\right) & \frac{sc}{L}\left(\frac{1}{\lambda^2}-12\right) & -6s & -\frac{1}{L}\left(\frac{c^2}{\lambda^2}+12s^2\right) & -\frac{sc}{L}\left(\frac{1}{\lambda^2}-12\right) & -6s \\ \frac{sc}{L}\left(\frac{1}{\lambda^2}-12\right) & \frac{1}{L}\left(\frac{s^2}{\lambda^2}+12c^2\right) & 6c & -\frac{sc}{L}\left(\frac{1}{\lambda^2}-12\right) & -\frac{1}{L}\left(\frac{s^2}{\lambda^2}+12c^2\right) & 6c \\ -6s & 6c & 4L & -6s & -6c & 2L \\ -\frac{1}{L}\left(\frac{c^2}{\lambda^2}+12s^2\right) & -\frac{sc}{L}\left(\frac{1}{\lambda^2}-12\right) & -6s & \frac{1}{L}\left(\frac{c^2}{\lambda^2}+12s^2\right) & \frac{sc}{L}\left(\frac{1}{\lambda^2}-12\right) & 6s \\ -\frac{sc}{L}\left(\frac{1}{\lambda^2}-12\right) & -\frac{1}{L}\left(\frac{s^2}{\lambda^2}+12c^2\right) & -6c & \frac{sc}{L}\left(\frac{1}{\lambda^2}-12\right) & \frac{1}{L}\left(\frac{s^2}{\lambda^2}+12c^2\right) & -6c \\ -6s & 6c & 2L & 6s & -6c & 4L \end{array}\right] \begin{array}{c} F_{ix} \\ F_{iy} \\ M_i \\ F_{jx} \\ F_{jy} \\ M_j \end{array} \end{array} \tag{200}$$

$$\lambda^2 = \frac{J_z}{AL^2} \qquad c = \cos\alpha \qquad s = \sin\alpha$$

3.3. Static reduction of the stiffness matrix

Frame elements are not always joined at a node ensuring the agreement of all nodal displacements and displacements in the bar section at this node. Articulated joints shown in Figure 28are examples of such incomplete connections.

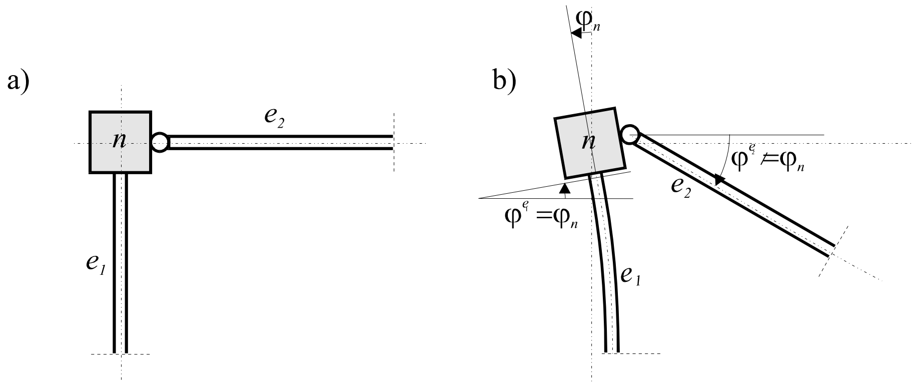


Figure 28. The element joint scheme with one element able to rotate (an articulated joint).

At this joint, the angle of the nodal rotation does not influence the rotation of the element section of a node. The latter can rotate independently of the node (the element e_2 in Figure 28).

We determine the unknown angle of the rotation of such an element using an additional equation which is given by the equilibrium condition of moments in a joint. Hence we can reduce the number of degrees of freedom of the element because the additional equilibrium condition allows us to eliminate one displacement from the set of equations. We will show the way to eliminate the degree of freedom using the example of two types of connections of an element with a node.

Example No 1 - articulated connection (Figure 29).

Figure 29. The element with the articulated *i* node joint.

Additional equilibrium condition of a section at the node i :

$$M_i = 0, \tag{201}$$

leads, after considering Eqn. (175), (182)and(197), to conditions:

$$\frac{EJ_z}{L}\left[6\frac{u_{iy}}{L}+4\varphi_i-6\frac{u_{jy}}{L}+2\varphi_j\right]=0, \tag{202}$$

and thus we calculate the required value of the rotation angle of the section at the node i:

$$\varphi_i = -\frac{3}{2}\frac{u_{iy}}{L} + \frac{3}{2}\frac{u_{jy}}{L} - \frac{1}{2}\varphi_j \,. \tag{203}$$

After putting this result into Eqn. (182) and taking into consideration matrix (197) we obtain:

$$F_{iy} = \frac{EJ_z}{L^2}\left[12\frac{u_{iy}}{L} + 6\left(-\frac{3}{2}\frac{u_{iy}}{L} + \frac{3}{2}\frac{u_{jy}}{L} - \frac{1}{2}\varphi_j\right) - 12\frac{u_{jy}}{L} + 6\varphi_j\right] =$$

$$= \frac{EJ_z}{L^2}\left[3\frac{u_{iy}}{L} - 3\frac{u_{jy}}{L} + 3\varphi_j\right],$$

$$F_{jy} = \frac{EJ_z}{L^2}\left[-12\frac{u_{iy}}{L} - 6\left(-\frac{3}{2}\frac{u_{iy}}{L} + \frac{3}{2}\frac{u_{jy}}{L} - \frac{1}{2}\varphi_j\right) + 12\frac{u_{jy}}{L} - 6\varphi_j\right] =$$

$$= \frac{EJ_z}{L^2}\left[-3\frac{u_{iy}}{L} + 3\frac{u_{jy}}{L} - 3\varphi_j\right], \tag{204}$$

$$M_j = \frac{EJ_z}{L}\left[6\frac{u_{iy}}{L} + 2\left(-\frac{3}{2}\frac{u_{iy}}{L} + \frac{3}{2}\frac{u_{jy}}{L} - \frac{1}{2}\varphi_j\right) - 6\frac{u_{jy}}{L} + 4\varphi_j\right] =$$

$$= \frac{EJ_z}{L}\left[3\frac{u_{iy}}{L} - 3\frac{u_{jy}}{L} + 3\varphi_j\right],$$

and hence the new stiffness matrix of an element with the joint at the node *i*:

$$\mathbf{K'}^{e(3,i)} = \begin{bmatrix} \frac{EA}{L} & 0 & & -\frac{EA}{L} & 0 & 0 \\ 0 & 3\frac{EJ_z}{L^3} & & 0 & -3\frac{EJ_z}{L^3} & 3\frac{EJ_z}{L^2} \\ & & & & & \\ -\frac{EA}{L} & 0 & & \frac{EA}{L} & 0 & 0 \\ 0 & -3\frac{EJ_z}{L^3} & & 0 & 3\frac{EJ_z}{L^3} & -3\frac{EJ_z}{L^2} \\ 0 & 3\frac{EJ_z}{L^2} & & 0 & -3\frac{EJ_z}{L^2} & 3\frac{EJ_z}{L} \end{bmatrix}. \tag{205}$$

Superscripts $(3,i)$ in the notation of the stiffness matrix indicate that the third degree of freedom is eliminated at the first node.

Example No 2 - moveable connection (
Figure 30)

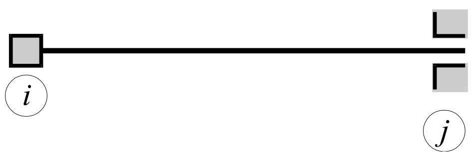


Figure 30. The element with the *j* node translation possibility.

Here the additional condition is the disappearance of the axial force at the node *j*:

$F_{jx} = 0,$ (206)

which after analogous transformations leads to the equation:

$F_{ix} = 0,$ (207)

and it does not change the relations for the remaining nodal forces.

The stiffness matrix of such an element takes the following form:

$$\mathbf{K}'^{e(1,j)} = \begin{bmatrix} & & & & & \\ & 12\frac{EJ_z}{L^3} & 6\frac{EJ_z}{L^2} & & -12\frac{EJ_z}{L^3} & 6\frac{EJ_z}{L^2} \\ & 6\frac{EJ_z}{L^2} & 4\frac{EJ_z}{L} & & -6\frac{EJ_z}{L^2} & 2\frac{EJ_z}{L} \\ & & & & & \\ & -12\frac{EJ_z}{L^3} & -6\frac{EJ_z}{L^2} & & 12\frac{EJ_z}{L^3} & -6\frac{EJ_z}{L^2} \\ & 6\frac{EJ_z}{L^2} & 2\frac{EJ_z}{L} & & -6\frac{EJ_z}{L^2} & 4\frac{EJ_z}{L} \end{bmatrix}. \tag{208}$$

Superscripts (1,*j*) in the notation of this matrix indicate that the first degree of freedom at the last node of an element has been eliminated.

The above process is called the static reduction of a stiffness matrix. Now we will give the matrix notation of an operation leading to a reduced stiffness matrix. For the sake of simplicity, we assume that the last degree of freedom of an element is the eliminated degree of freedom. Nodal forces, nodal displacements vectors and the stiffness matrix are divided into blocks:

$$\begin{bmatrix} \mathbf{f}_1 \\ \hline \mathbf{f}_0 \end{bmatrix} = \left[\begin{array}{c|c} \mathbf{K}_{11} & \mathbf{K}_{10} \\ \hline \mathbf{K}_{01} & \mathbf{K}_{00} \end{array}\right] \begin{bmatrix} \mathbf{u}_1 \\ \hline \mathbf{u}_0 \end{bmatrix}, \tag{209}$$

where according to the symmetry of the matrix we have

$\mathbf{K}_{11} = \mathbf{K}_{11}^{\mathsf{T}}$, $\mathbf{K}_{01} = \mathbf{K}_{10}^{\mathsf{T}}$,

$\mathbf{K}_{00}$ is the matrix with dimension 1×1 and thus it is a scalar, the blocks $\mathbf{f}_0$ and $\mathbf{u}_0$ are also scalars. The results of the multiplication of matrix blocks Eqn. (209)are:

a) $\mathbf{f}_1 = \mathbf{K}_{11}\mathbf{u}_1 + \mathbf{K}_{10}\mathbf{u}_0$,

b) $\mathbf{f}_0 = \mathbf{K}_{01}\mathbf{u}_1 + \mathbf{K}_{00}\mathbf{u}_0 = 0$ – scalar. (210)

From Eqn. (210b) we calculate

$$\mathbf{u}_0 = -\mathbf{K}_{00}^{-1}\,\mathbf{K}_{01}\,\mathbf{u}_1\,, \tag{211}$$

and after inserting this relation into (210a) we obtain

$$\mathbf{f}_1 = \mathbf{K}_{11}\mathbf{u}_1 - \mathbf{K}_{10}\mathbf{K}_{00}^{-1}\mathbf{K}_{01}\mathbf{u}_1\,, \tag{212}$$

or

$$\mathbf{f}_1 = \mathbf{K}''\mathbf{u}_1\,, \tag{213}$$

where

$$\mathbf{K}'' = \mathbf{K}_{11} - \mathbf{K}_{10}\mathbf{K}_{00}^{-1}\mathbf{K}_{10}^{\mathsf{T}}\,, \tag{214}$$

is the condensed element stiffness matrix.

Vector $\mathbf{f}_1$ of an element load still remains to be determined. We obtain it by composing both the load vector $\mathbf{f}^{o}$ of an element with rigid connections with nodes and the vector $\mathbf{f}^{u}$ of the load caused by displacements of nodes free from constraints

$$\mathbf{f} = \mathbf{f}^{o} - \mathbf{f}^{u} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_0 \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1^{o} \\ \mathbf{f}_0^{o} \end{bmatrix} - \begin{bmatrix} \mathbf{f}_1^{u} \\ \mathbf{f}_0^{u} \end{bmatrix}. \tag{215}$$

Since

$$\mathbf{f}_0 = \mathbf{f}_0^{o} - \mathbf{f}_0^{u} = 0\,, \tag{216}$$

then

$$\mathbf{f}_0^{u} = \mathbf{f}_0^{o} = \mathbf{K}_{01}^{o}\mathbf{u}_1 + \mathbf{K}_{00}^{o}\mathbf{u}_0\,, \tag{217}$$

and hence

$$\mathbf{u}_0 = \left(\mathbf{K}_{00}^{o}\right)^{-1}\mathbf{f}_0^{o}\,, \tag{218}$$

because other displacements contained in $\mathbf{u}_1$ are equal to zero. Finally, we obtain

$$\mathbf{f}_1 = \mathbf{f}_1^{o} - \mathbf{K}_{10}^{o}\left(\mathbf{K}_{00}^{o}\right)^{-1}\mathbf{f}_0^{o}\,. \tag{219}$$

In this way, we can eliminate any degree of freedom but it requires some more complex transformations. We leave this problem to be solved by the reader.

3.4. Boundary conditions of plane frame structures

Supports for plane frames include articulated and fixed supports all listed in Chapter 2. The latter ones prevent the rotation of a support node. Symbolic notation of these supports and the boundary conditions describing them are shown in Figure 31.

Considering boundary conditions requires the modification of a global stiffness matrix of a structure and it is done identically as for a plane truss (Sec.2.6), thus, we will not describe the way of modifying this matrix here. A whole range of other supports such as moveable skew supports and elastic supports considered analogously to supports of trusses described in Chapter 2 is also possible.

As a general method of consideration of non-typical supports, we propose to consider the use of suitable boundary elements instead of these supports. We will discuss this in the next section.

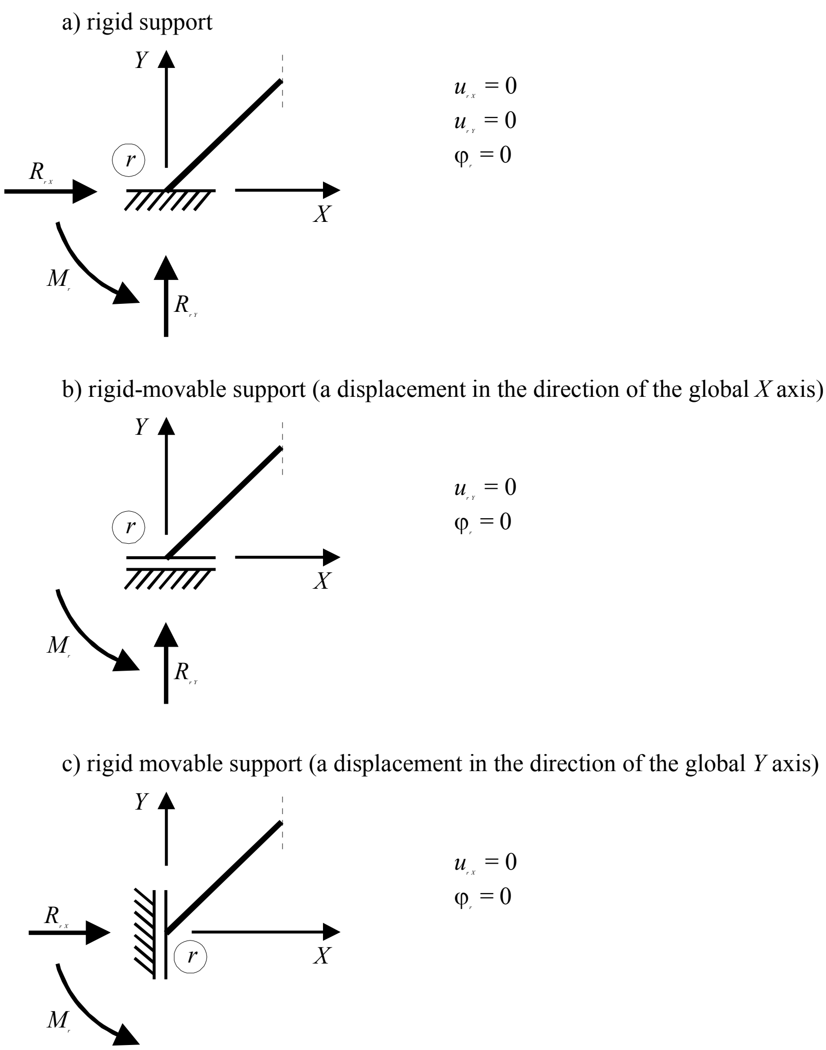


Figure 31. Plane frame support types.

3.5. Boundary elements of 2D frames

Introducing a boundary element is a convenient way to avoid problems connected with the consideration of different, non-typical boundary conditions. It allows, in fact, us to model fixed and fixed-movable supports with approximate exactness and to substitute elastic supports.

Now we will present a single elastic support inclined at some angle. The scheme of this element and notations used are shown inFigure 32.

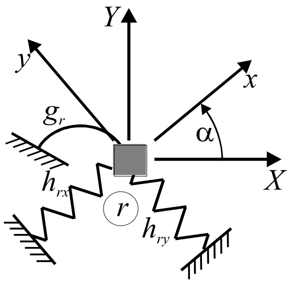


Figure 32. The plane frame elastic support scheme.

Stiffness of springs: h_{rx} and h_{ry} are forces which should be applied to their ends in order to induce unitary extensions. Rotation stiffness of a support g_r is a moment necessary to induce the rotation of the node r equal to one radian.

The stiffness matrix of such an element in the local coordinate system has the form:

$$\mathbf{K}^{b} = \begin{bmatrix} h_{rx} & 0 & 0 \\ 0 & h_{ry} & 0 \\ 0 & 0 & g_r \end{bmatrix}. \tag{220}$$

Its transformation to the global system is done analogously to the case of normal frame or truss elements except that it concerns one node only Eqn. (95). The rotation matrix is given by Eqn. (188). Hence we can write the equation transforming the matrix $\mathbf{K}'^{b}$ to the global system:

$$\mathbf{K}^{b} = \mathbf{R}_r \mathbf{K}'^{b} \mathbf{R}_r^{\mathsf{T}}. \tag{221}$$

After taking into consideration Eqn. (188)and (220)we obtain

$$\mathbf{K}^b = \begin{bmatrix} c^2 h_{rx} + s^2 h_{ry} & sc\left(h_{rx} - h_{ry}\right) & 0 \\ sc\left(h_{rx} - h_{ry}\right) & c^2 h_{rx} + s^2 h_{ry} & 0 \\ 0 & 0 & g_r \end{bmatrix}, \tag{222}$$

where $s = \sin\alpha$, $c = \cos\alpha$.

If we model flexible supports we ought to assume high stiffness of a suitable spring. In most cases, stiffness of the order of 1×10^{30} assures similarity between results obtained with this method and the results obtained with the exact method.

3.6. Internal forces due to a static load

The variety of loads which can act on a frame structure is considerably greater than it was in the case of a truss. Frame elements can be affected by concentrated (forces, moments), distributed (pressure, moment loads) and temperature loads. The formulation of equilibrium equations requires substitution of internode loads for an equivalent set of concentrated forces and moments acting on nodes. The way of reducing these loads will be the subject of our discussion in this section.

Eqn. (190) and (195)define displacements of an element bending in the direction of the y axis of the global system. After adding the equations describing the displacements in an axial direction, we obtain relations defining the displacements vector for any point between nodes

$$\mathbf{u}(x) = \begin{bmatrix} u_x(x) \\ u_y(x) \\ \varphi(x) \end{bmatrix} = \mathbf{N}\mathbf{u}^e , \tag{223}$$

where $\mathbf{N}$ is the rectangular matrix of shape functions. It contains two blocks: $\mathbf{N}_i(x)$ - matrix of the shape functions for the first node and $\mathbf{N}_j(x)$ - matrix of the shape functions for the last node.

$$\mathbf{N}(x) = \left[\mathbf{N}_i(x) \quad \mathbf{N}_j(x)\right]. \tag{224}$$

We can obtain both matrices from Eqn. (190) and (71):

$$\mathbf{N}_i(x) = \begin{bmatrix} \omega_1(\xi) & 0 & 0 \\ 0 & \omega_3(\xi) & L\omega_5(\xi) \\ 0 & \frac{1}{L}\omega_3{}'(\xi) & \omega_5{}'(\xi) \end{bmatrix}, \tag{225}$$

$$\mathbf{N}_j(x)=\begin{bmatrix} \omega_2(\xi) & 0 & 0 \\ 0 & \omega_4(\xi) & L\omega_6(\xi) \\ 0 & \frac{1}{L}\omega_4{}'(\xi) & \omega_6{}'(\xi) \end{bmatrix},$$

where non-dimensional displacement functions $\omega_i(\xi)$ (i = 1,2 ... 6)) and their derivatives $\omega_i{}'(\xi)$, $\omega_i{}''(\xi)$ are surveyed in Table 4. The convenient non-dimensional coordinate $\xi = x/L$ is introduced here.

Let us consider now the bar (an element) of a plane frame loaded with static loads (Figure 33).

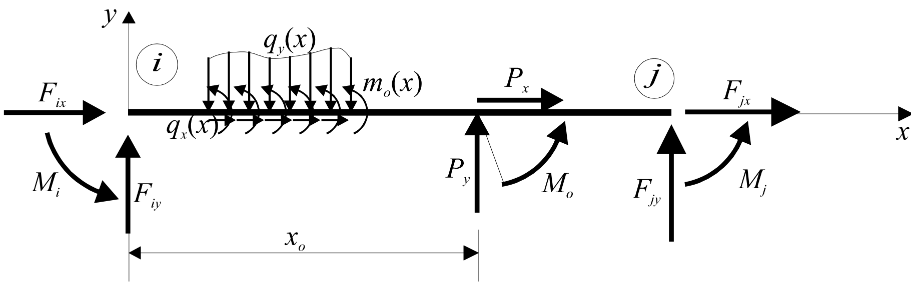


Figure 33. The plane frame element loaded with static loads.

We will find nodal forces $\mathbf{f}^e$ by making use of conditions of element equilibrium. We will use the principle of virtual work here:

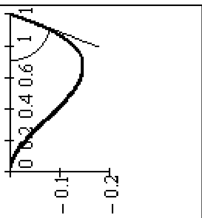
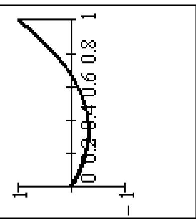
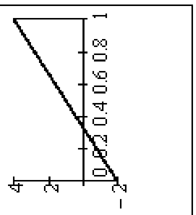
$$L_n = \left(\mathbf{f}'^e\right)^{\mathrm{T}}\mathbf{u}^e \tag{226a}$$

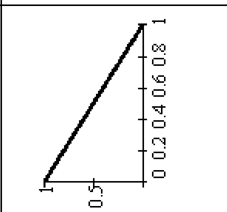
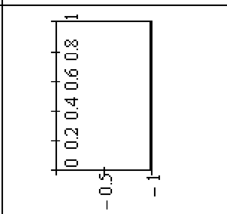
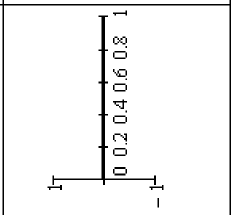
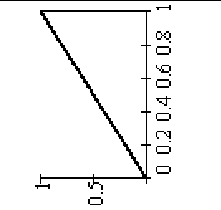
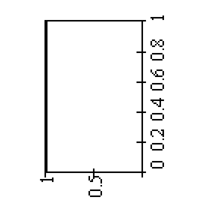
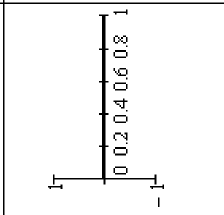
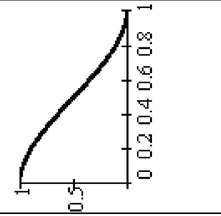
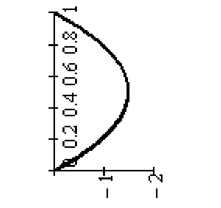
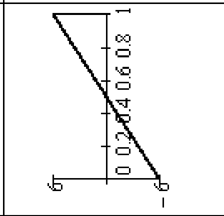
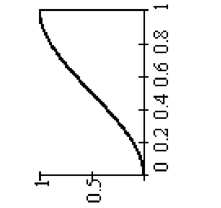
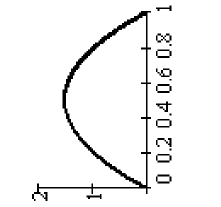
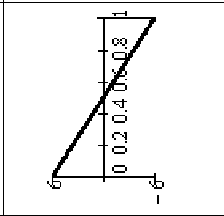
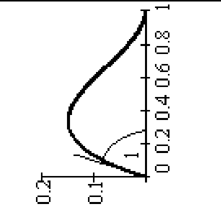
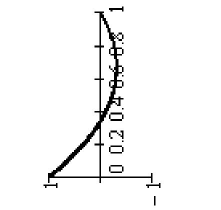
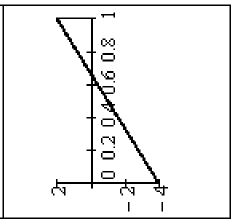
where L_n is the work of nodal forces,

$$L_z = \int_0^L \left[q_y(x)u_y(x) + q_x(x)u_x(x) + m_o(x)\varphi(x)\right]dx \tag{226b}$$

where L_z is the work of external forces (static loads).

Table 4.Non-dimensional displacement functions.

6	$-\xi^2(1-\xi)$		$-\xi(2-3\xi)$		$-2+6\xi$	

Nr	ω_i	ω_i graph	ω_i'	ω_i' graph	ω_i''	ω_i'' graph
1	$1-\xi$		-1		0	
2	ξ		1		0	
3	$1-3\xi^2+2\xi^3$		$-6\xi(1-\xi)$		$-6+12\xi$	
4	$\xi^2(3-2\xi)$		$6\xi(1-\xi)$		$6-12\xi$	
5	$\xi(1-2\xi+\xi^3)$		$1-4\xi+3\xi^2$		$-4+6\xi$	

Concentrated forces and moments can also be analysed by describing them in the following way:

$$q(x)=\delta(x-x_o)P, \qquad M_o=\delta(x-x_o)M_o, \tag{227}$$

where $\delta(x_o)$ is Dirac's delta defined as (comp. Nowacki (1979))

$$\delta(x-x_o)=0, \text{ while } x<x_o;$$

$$\delta(x-x_o)\to\infty, \text{ while } x=x_o; \tag{228}$$

$$\delta(x-x_o)=0, \text{ while } x>x_o$$

and $\int\limits_{-\infty}^{+\infty} \delta(x - x_o)dx = 1$.

The element equilibrium is maintain when L_n+L_z= 0, which means

$$\left(\mathbf{f}'^{e}\right)^{\mathsf{T}} \mathbf{u}^{e} = -\int\limits_{0}^{L} [\mathbf{q}(x)]^{\mathsf{T}} \mathbf{u}(x) dx \,, \tag{229}$$

where $\mathbf{q}(x)$ is the vector of external loads:

$$\mathbf{q}(x) = \begin{bmatrix} q_x(x) \\ q_y(x) \\ m_o(x) \end{bmatrix}. \tag{230}$$

Putting the expression describing the element displacements vector Eqn. (223)into (229)we obtain relations:

$$\left(\mathbf{f}'^{e}\right)^{\mathsf{T}} \mathbf{u}^{e} = -\int\limits_{0}^{L} \mathbf{q}^{\mathsf{T}} \mathbf{N} \mathbf{u}^{e} dx \,, \tag{231}$$

$$\left(\mathbf{f}'^{e}\right)^{\mathsf{T}} = -\int\limits_{0}^{L} \mathbf{N}\mathbf{q} \, dx \,, \tag{232}$$

which enables us to replace loads acting on elements by loads acting on nodes. It should be noted here that there are forces acting on the nodes in the equilibrium equations and that these forces act against those acting on the element (comp. Figure. 18) thus, they should be subtracted from the nodal forces vector of the structure.

We check the effectiveness of Eqn. (232) for three simple examples when:

1. the load with a concentrated force is applied to the centre of an element,
2. the load with a concentrated moment,
3. the distributed load which is constant for the whole element.

Example No 1.

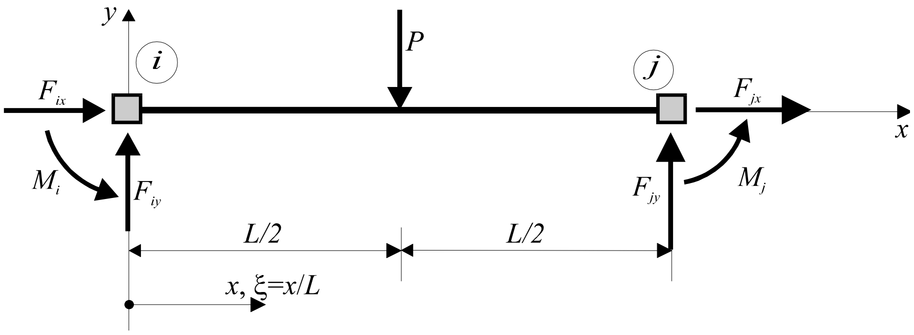


Figure 34. The frame element loaded with a concentrated force.

We introduce a non-dimensional coordinate $\xi = x/L$ to make the calculations more convenient and write the concentrated force as follows:

$$\mathbf{q}(\xi) = \begin{bmatrix} 0 \\ P\delta(\xi - 0.5) \\ 0 \end{bmatrix}$$

and after putting it into Eqn. (232) we obtain

$$\mathbf{f}'^{e} = -L^2 \int_0^1 \begin{bmatrix} (\mathbf{N}_i)^{\mathrm{T}} \\ (\mathbf{N}_j)^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} 0 \\ -P\delta(\xi - 0.5) \\ 0 \end{bmatrix} d\xi = P\int_0^1 \begin{bmatrix} 0 \\ \delta(\xi-0.5)\omega_3(\xi) \\ L\delta(\xi-0.5)\omega_5(\xi) \\ 0 \\ \delta(\xi-0.5)\omega_4(\xi) \\ L\delta(\xi-0.5)\omega_6(\xi) \end{bmatrix} d\xi = P\begin{bmatrix} 0 \\ \omega_3(0.5) \\ L\omega_5(0.5) \\ 0 \\ \omega_4(0.5) \\ L\omega_6(0.5) \end{bmatrix} = P\begin{bmatrix} 0 \\ 1/2 \\ L/8 \\ 0 \\ 1/2 \\ -L/8 \end{bmatrix}$$

which means that

$$F_{ix} = 0, \qquad F_{iy} = \frac{1}{2}P, \qquad M_i = \frac{1}{8}PL,$$

$$F_{jx} = 0, \qquad F_{jy} = \frac{1}{2}P, \qquad M_j = -\frac{1}{8}PL.$$

Example No 2.

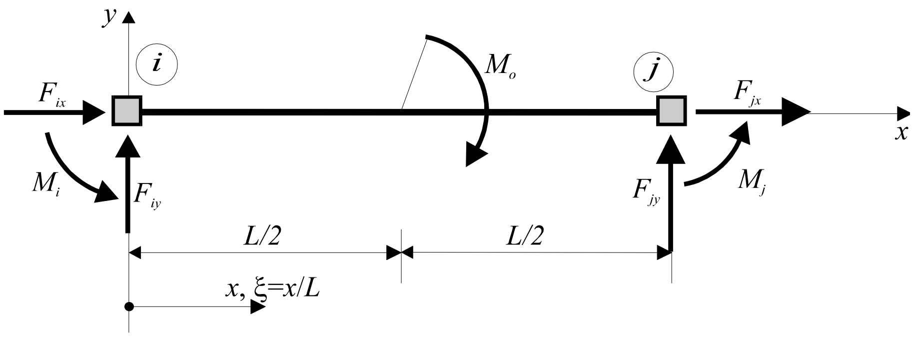


Figure 35. The frame element loaded with a concentrated moment.

We write the concentrated moment applied to the centre of an element by using Dirac's delta:

$$\mathbf{q}(\xi) = \begin{bmatrix} 0 \\ 0 \\ -\dfrac{M_o}{L}\,\delta(\xi - 0.5) \end{bmatrix}$$

After inserting the load vector into Eqn. (232), we obtain

$$\mathbf{f}'^e = \frac{M_o}{L}\int_0^1 \begin{bmatrix} (\mathbf{N}_i)^{\mathsf{T}} \\ (\mathbf{N}_j)^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \delta(\xi - 0.5) \end{bmatrix} d\xi = \frac{M_o}{L}\int_0^1 \begin{bmatrix} 0 \\ \omega_3'(\xi)\delta(\xi-0.5)/L \\ \omega_5'(\xi)\delta(\xi-0.5) \\ 0 \\ \omega_4'(\xi)\delta(\xi-0.5)/L \\ \omega_6'(\xi)\delta(\xi-0.5) \end{bmatrix} d\xi = M_o \begin{bmatrix} 0 \\ -\dfrac{3}{2L} \\ -\dfrac{1}{4} \\ 0 \\ \dfrac{3}{2L} \\ -\dfrac{1}{4} \end{bmatrix}$$

which means that

$$F_{ix} = 0, \qquad F_{iy} = -\frac{3}{2L}M_o, \qquad M_i = -\frac{1}{4}M_o,$$

$$F_{jx} = 0, \qquad F_{jy} = \frac{3}{2L}M_o, \qquad M_j = -\frac{1}{4}M_o.$$

Example No 3.

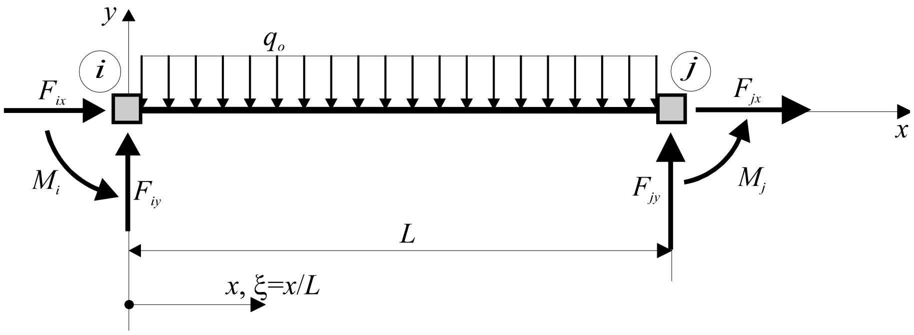


Figure 36. The frame element loaded with a uniformly distributed load.

The continuous load uniformly distributed on the whole length of an element gives a load vector:

$$\mathbf{q}(\xi) = q_o \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}.$$

After inserting the vector $\mathbf{q}(\xi)$ into Eqn. (232), we obtain the equation:

$$\mathbf{f}'^{e} = q_o L \int_0^1 \begin{bmatrix} 0 \\ \omega_3(\xi) \\ L\omega_5(\xi) \\ 0 \\ \omega_4(\xi) \\ L\omega_6(\xi) \end{bmatrix} d\xi = q_o L \begin{bmatrix} 0 \\ 1/2 \\ L/12 \\ 0 \\ 1/2 \\ -L/12 \end{bmatrix},$$

which means that

$$F_{ix} = 0, \qquad F_{iy} = \frac{1}{2} q_o L, \quad M_i = \frac{1}{12} q_o L,$$

$$F_{jx} = 0, \qquad F_{jy} = \frac{1}{2} q_o L, \quad M_j = -\frac{1}{12} q_o L.$$

3.7. Forces caused by a temperature load

The action of a temperature on frame elements can cause flexion. This happens when the temperature field is not homogeneous in the cross section. In the case of a truss, the flexion of bars did not cause increasing nodal forces because truss elements are connected by means of jointed nodes. Bars of frame structures can make a node rotate, hence we have to determine forces at the node in the element undergoing the action of the non-uniform temperature field.

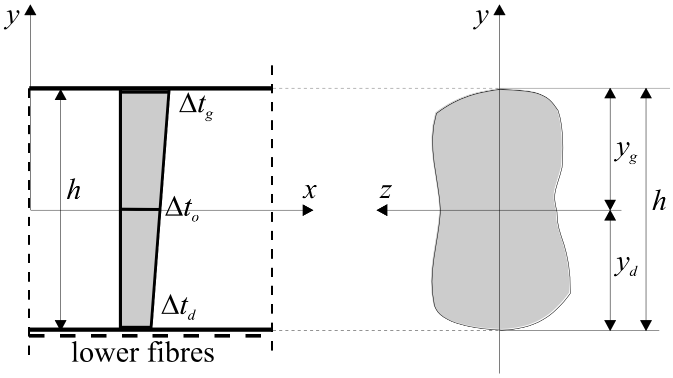


Figure 37. The temperature distribution in the element cross section.

Let us consider an element of which the upper fibres are affected by an increase in a temperature Δt_g, and the lower fibres are affected by an increase in a temperature Δt_d (Figure 37). The temperature field can be written as follows:

$$\Delta t(x,y) = \Delta t_o(x) + \frac{y}{h}\Delta t_h(x) \quad , \tag{233}$$

where $\Delta t_o = \frac{1}{h}\left[\Delta t_d y_g + \Delta t_g y_d\right]$ is the increase in the temperature of the middle fibres, $\Delta t_h = \Delta t_g - \Delta t_d$ is the difference of temperatures between extreme fibres, h is the height of the cross section, y_d is the distance between the centre of gravity and the lower fibres, y_g is the distance between the centre of gravity and the upper fibres.

Strains of the element fibres induced by the temperature field are equal to

$$\varepsilon_t(y) = \alpha_t \Delta t(y) = \alpha_t\left(\Delta t_o + \Delta t_h \frac{y}{h}\right) \quad , \tag{234}$$

where α_t is the expansion coefficient of the material.

If bars cannot deform freely, then stresses rise inside them:

$$\sigma_x = -E\varepsilon_t = -\alpha_t E\left(\Delta t_o + \Delta t_h \frac{y}{h}\right) \quad , \tag{235}$$

which the internal forces result from:

$$N = \int_A \sigma_x dA = -\alpha_t E\left(\Delta t_o \int_A dA + \frac{\Delta t_h}{h}\int_A y dA\right). \tag{236}$$

Since the second integral occurring in Eqn. (236) is the static moment with regard to the z axis which crosses the centre of gravity, this moment has to be equal to zero. Thus, we obtain

$$N_t(x) = -\alpha_t \Delta t_o(x) EA\,, \tag{237}$$

like in the case of a truss element.

The second internal force caused by temperature stresses is the bending moment:

$$M_t(x) = \int_A -\sigma_x(x) y dA = \alpha_t E\left[\Delta t_o(x)\int_A ydA + \frac{\Delta t_h}{h}\int_A y^2 dA\right]. \tag{238}$$

The first integral in the above equation has to be equal to zero similarly to Eqn. (236), and the second one is the moment of inertia of the cross section calculated with regard to the middle axis. Thus, we can write an equation describing the bending moment due to temperature stresses as

$$M_t(x) = \frac{\alpha_t \Delta t_h(x)}{h} EJ_z\,, \tag{239}$$

where $J_z = \int_A y^2 dA$ is the moment of inertia of the element section with regard to the z axis crossing the centre of gravity of the section.

We calculate forces at nodes making use of the principle of virtual work just as we did in Sec.4.6:

$$\left(\mathbf{u}^e\right)^{\mathrm{T}} \mathbf{f}'^{et} = \int_0^L [\varepsilon(x)]^{\mathrm{T}} \mathbf{t}_t dx\,, \tag{240}$$

where

$$\mathbf{t}_t = \begin{bmatrix} N_t(x) \\ 0 \\ M_t(x) \end{bmatrix} \tag{241}$$

is the vector of the internal forces induced by a temperature. The zero value of the expression in the second row of the vector comes from the fact that the temperature does not cause shearing forces in the elements, $\boldsymbol{\varepsilon}(x)$ is the vector of displacements gradients:

$$\boldsymbol{\varepsilon}(x) = \begin{bmatrix} \dfrac{du_x}{dx} \\ \dfrac{du_y}{dy} \\ \dfrac{d\varphi}{dx} \end{bmatrix} = \mathbf{B}\mathbf{u}^e \, , \tag{242}$$

B is the matrix of derivatives of shape functions:

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_i & \mathbf{B}_j \end{bmatrix}. \tag{243}$$

On the basis of Eqn. (225) we calculate

$$\mathbf{B}_i(x) = \begin{bmatrix} \frac{1}{L}\omega_1'(\xi) & 0 & 0 \\ 0 & \frac{1}{L}\omega_3'(\xi) & \omega_5'(\xi) \\ 0 & \frac{1}{L^2}\omega_3''(\xi) & \frac{1}{L}\omega_5''(\xi) \end{bmatrix},$$

$$\mathbf{B}_j(x) = \begin{bmatrix} \frac{1}{L}\omega_2'(\xi) & 0 & 0 \\ 0 & \frac{1}{L}\omega_4'(\xi) & \omega_6'(\xi) \\ 0 & \frac{1}{L^2}\omega_4''(\xi) & \frac{1}{L}\omega_6''(\xi) \end{bmatrix}, \tag{244}$$

where $\omega_i(\xi)$, $\omega_i^{'}(\xi)$, $\omega_i^{''}(\xi)$ (i = 1,2 ... 6)) are nondimensional functions given inTable 4.

On the basis of Eqn. (240) we calculate components of the nodal forces vector:

$$\mathbf{f}'^{et} = \int_0^L \mathbf{B}^{\mathsf{T}} \mathbf{t}_t dx \, . \tag{245}$$

After inserting matrix Eqn. (244)into Eqn. (245), we obtain

$$\mathbf{f}'^{et} = \alpha_t E \begin{bmatrix} -A\int_{\xi_1}^{\xi_2} \omega_1'(\xi)\Delta t_o(\xi)d\xi \\ \dfrac{J_z}{hL}\int_{\xi_1}^{\xi_2} \omega_3''(\xi)\Delta t_h(\xi)d\xi \\ \dfrac{J_z}{h}\int_{\xi_1}^{\xi_2} \omega_5''(\xi)\Delta t_h(\xi)d\xi \\ -A\int_{\xi_1}^{\xi_2} \omega_2'(\xi)\Delta t_o(\xi)d\xi \\ \dfrac{J_z}{hL}\int_{\xi_1}^{\xi_2} \omega_4''(\xi)\Delta t_h(\xi)d\xi \\ \dfrac{J_z}{h}\int_{\xi_1}^{\xi_2} \omega_6''(\xi)\Delta t_h(\xi)d\xi \end{bmatrix}, \tag{246}$$

where ξ_1 and ξ_2 are non-dimensional coordinates at both the beginning and end of the action interval of the temperature load (Figure 38).

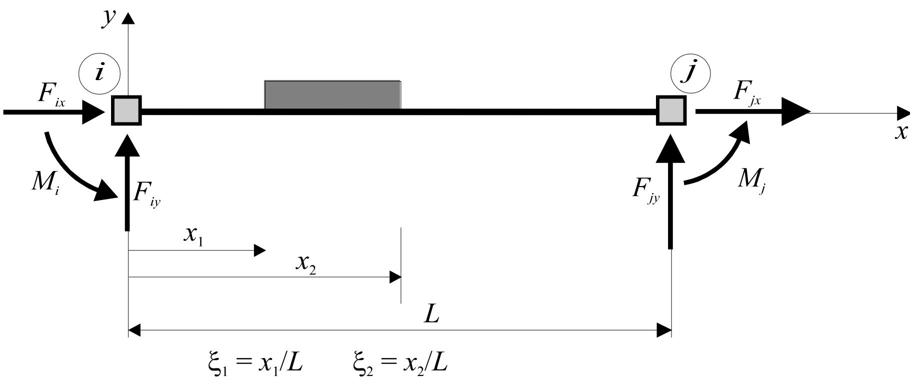


Figure 38. The temperature loaded frame element.

In the case when the temperature load is constant and occurs along the whole length of the element, we obtain the following equation from Eqn. (225):

$$\mathbf{f}'^{et} = \alpha_t E \begin{bmatrix} A\Delta t_o \\ 0 \\ -\dfrac{J_z\Delta t_h}{h} \\ -A\Delta t_o \\ 0 \\ \dfrac{J_z\Delta t_h}{h} \end{bmatrix}. \tag{247}$$

Both Eqn. (225)and (226) describe internal forces acting on the element. So when we form the load vector of a structure we should subtract components of this vector from suitable components of the global vector.

Statics of a 3D frame system

A three-dimensional frame structure is the most general type of bar structures. Elements of a space frame can serve for modelling of all the previously described structures (2D and 3D trusses, 2D frames) and some others such as grillworks, beams broken in a plane and loaded perpendicularly to its plane, etc. A few examples of structures which cannot be modelled by elements presented so far but can only be modelled with the help of 3D frame elements are presented in Figure 39.

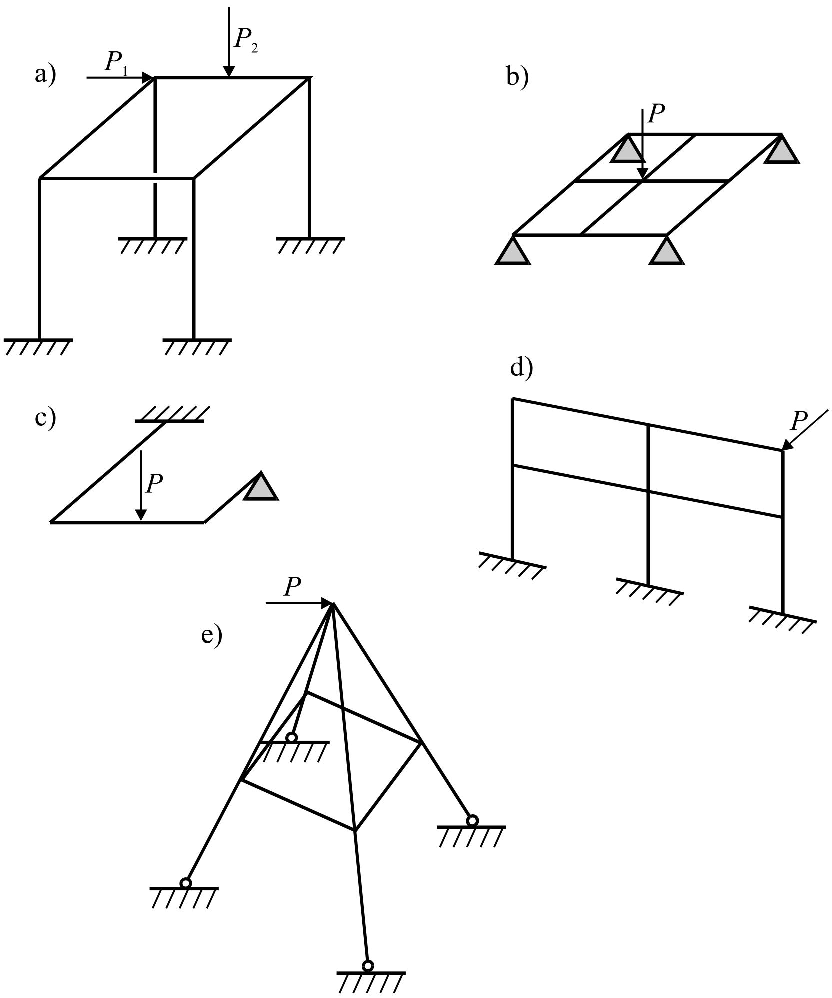


Figure 39. The 3D frame examples.

3.8. The element stiffness matrix of a 3D frame

Any node of a space structure has six degrees of freedom which means that it can submit to three independent displacements and three rotations. Hence a frame element has twelve degrees of freedom. Components of both nodal forces and displacements of the frame element are shown in Figure 40. The local coordinate system has to be chosen in such a way that axes *y* and *z* are the principal axes of a cross section because it simplifies the discussion of a bending of problem. Bending of such an element can be analysed as two independent phenomena of bending in planes *xy* and *xz*.

a) nodal displacements and forces

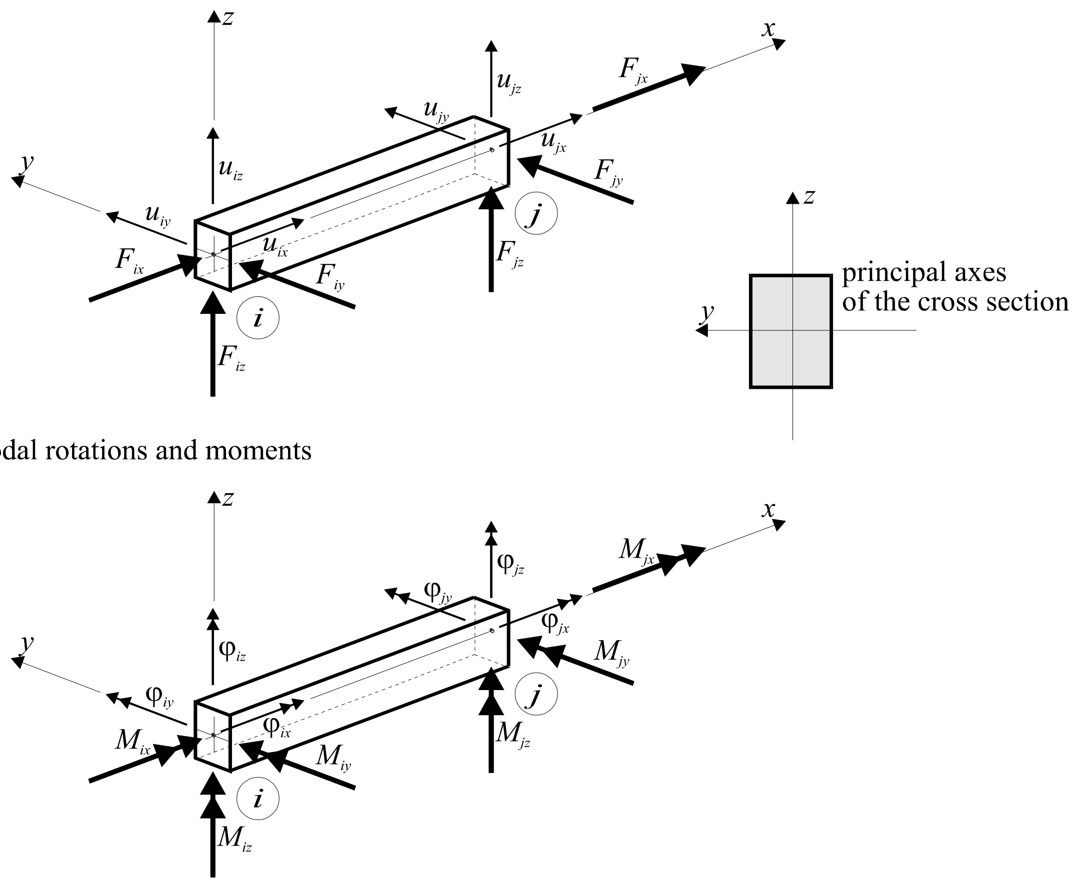


Figure 40. Nodal loads and displacements for the 3D frame element in the element local coordinate system.

Here we will present nodal displacements and forces similarly, that is, in the form of vectors (column matrices).

The nodal displacement vector of an element in the local system is:

$$\mathbf{u}'^{e} = \begin{bmatrix} \mathbf{u}'_i \\ \mathbf{u}'_j \end{bmatrix}, \tag{248}$$

where

$$\mathbf{u}'_i = \begin{bmatrix} u_{ix} \\ u_{iy} \\ u_{iz} \\ \varphi_{ix} \\ \varphi_{iy} \\ \varphi_{iz} \end{bmatrix}, \ \mathbf{u}'_j = \begin{bmatrix} u_{jx} \\ u_{jy} \\ u_{jz} \\ \varphi_{jx} \\ \varphi_{jy} \\ \varphi_{jz} \end{bmatrix}, \tag{249}$$

$\mathbf{u}'_i$ is the displacement vector of the node i in the local coordinate system, $\mathbf{u}'_j$ is the displacement vector of the node j in the local coordinate system.

The nodal force vector of an element in the local system is

$$\mathbf{f}'^e = \begin{bmatrix} \mathbf{f}'_i \\ \mathbf{f}'_j \end{bmatrix}, \tag{250}$$

where

$$\mathbf{f}'_i = \begin{bmatrix} F_{ix} \\ F_{iy} \\ F_{iz} \\ M_{ix} \\ M_{iy} \\ M_{iz} \end{bmatrix}, \quad \mathbf{f}'_j = \begin{bmatrix} F_{jx} \\ F_{jy} \\ F_{jz} \\ M_{jx} \\ M_{jy} \\ M_{jz} \end{bmatrix}, \tag{251}$$

$\mathbf{f}'_i$ is the force vector of the node i in the local coordinate system, $\mathbf{f}'_j$ is the force vector of the node j in the local coordinate system.

As usual we look for the relationship between nodal forces and displacements in the form:

$$\mathbf{f}'^e = \mathbf{K}'^e \, \mathbf{u}'^e, \tag{252}$$

where the stiffness matrix $\mathbf{K}'^e$ is a square and symmetric matrix with dimensions 12x12. Most components of this matrix can be calculated on the basis of the results obtained for a 2D frame in Chapter 4. Since the bending in principal planes of the cross section is independent, we will split the deformation of the element of a three-dimensional frame into a few simpler form:

- axial tension which is identical to that in a truss,
- bending in the *xz* plane which is similar to the states of a 2D frame; modifications concern the signs of internal forces,
- torsion.

Torsion of a frame element is a state which has not been described so far. The dependence between a nodal torsion moment and a torsion angle of an element is quite simple (comp. Jastrzębski et al. (1985)) and resembles the relation between an axial force and an element extension:

$$\frac{\Delta\varphi_x}{L} = \frac{M_x}{GC}, \tag{253}$$

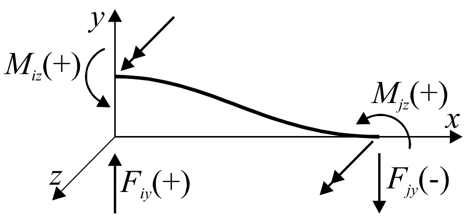
where $\Delta\varphi_x = \varphi_{jx} - \varphi_{ix}$ is the increase in the torsion angle due to the torsion moment M_x, $G = \frac{E}{2(1+\nu)}$ - is Kirchhoff's modulus and C is the torsional resistance characteristics.

The constant C has the dimension of a moment of inertia and is equal to the polar moment of inertia for circular-symmetric sections (comp. Jastrzębski et al. (1985)) but for other sections it should be calculated by use of quite complex methods (comp. Timoshenko and Goodier (1962)). The calculation method of this constant for a few popular cross sections in engineering practice is given in Appendix 3.

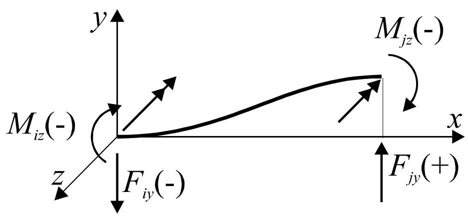
Eqn. (253) allows us to write the relation between the nodal rotations around the x axis and nodal torsion moments:

$$\begin{aligned} M_{ix} &= \frac{GC}{L}\left(\varphi_{ix} - \varphi_{jx}\right), \\ M_{jx} &= \frac{GC}{L}\left(-\varphi_{ix} + \varphi_{jx}\right). \end{aligned} \tag{254}$$

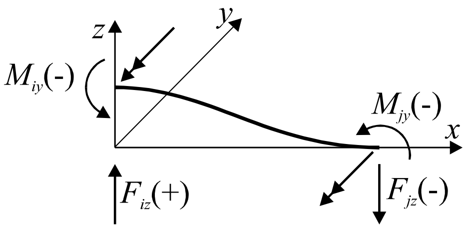
The above equations are the searched relation which allows us to write the element stiffness matrix. Senses of nodal forces caused by unitary nodal displacements, which allow us to determine signs of the expressions of the stiffness matrix, are shown in Figure 41.



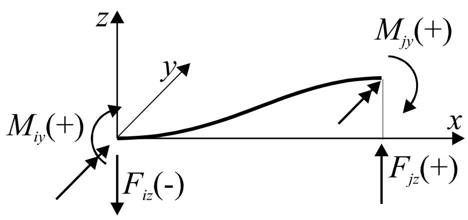
column No 2 - u_{iy}=1



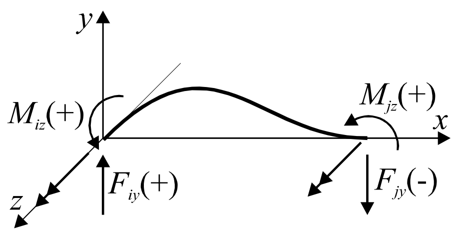
column No 8 - u_{jy}=1



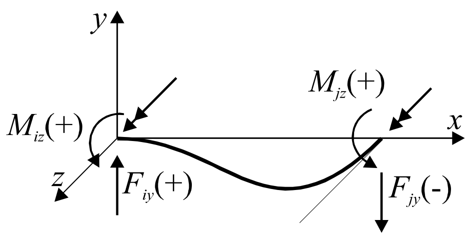
column No 3 - u_{iz}=1



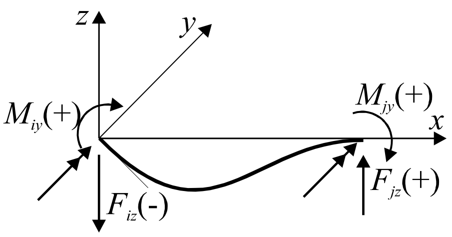
column No 9 - u_{jz}=1



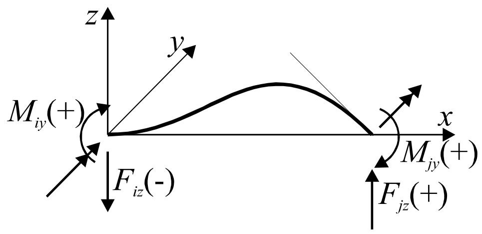
column No 6 - φ_{iz}=1



column No 12 - φ_{jz}=1



column No 5 - φ_{iy}=1



column No 11 - φ_{jy}=1

Figure 41. The signs of the nodal force vector caused by unitary nodal displacements.

The element stiffness matrix is presented by equation

$$
\begin{array}{c}
\begin{array}{cccccccccccc}
u_{ix} & u_{iy} & u_{iz} & \varphi_{ix} & \varphi_{iy} & \varphi_{iz} & u_{jx} & u_{jy} & u_{jz} & \varphi_{jx} & \varphi_{jy} & \varphi_{jz}
\end{array}
\\
\mathbf{K}'^{e} = \left[\begin{array}{cccccc|cccccc}
\frac{EA}{L} & & & & & & -\frac{EA}{L} & & & & & \\
& 12\frac{EJ_z}{L^3} & & & & 6\frac{EJ_z}{L^2} & & -12\frac{EJ_z}{L^3} & & & & 6\frac{EJ_z}{L^2} \\
& & 12\frac{EJ_y}{L^3} & & -6\frac{EJ_y}{L^2} & & & & -12\frac{EJ_y}{L^3} & & -6\frac{EJ_y}{L^2} & \\
& & & \frac{GC}{L} & & & & & & -\frac{GC}{L} & & \\
& & -6\frac{EJ_y}{L^2} & & 4\frac{EJ_y}{L^2} & & & & 6\frac{EJ_y}{L^2} & & 2\frac{EJ_y}{L^2} & \\
& 6\frac{EJ_z}{L^2} & & & & 4\frac{EJ_z}{L^2} & & -6\frac{EJ_z}{L^2} & & & & 2\frac{EJ_z}{L^2} \\
\hline
-\frac{EA}{L} & & & & & & \frac{EA}{L} & & & & & \\
& -12\frac{EJ_z}{L^3} & & & & -6\frac{EJ_z}{L^2} & & 12\frac{EJ_z}{L^3} & & & & -6\frac{EJ_z}{L^2} \\
& & -12\frac{EJ_y}{L^3} & & 6\frac{EJ_y}{L^2} & & & & 12\frac{EJ_y}{L^3} & & 6\frac{EJ_y}{L^2} & \\
& & & -\frac{GC}{L} & & & & & & \frac{GC}{L} & & \\
& & -6\frac{EJ_y}{L^2} & & 2\frac{EJ_y}{L^2} & & & & 6\frac{EJ_y}{L^2} & & 4\frac{EJ_y}{L^2} & \\
& 6\frac{EJ_z}{L^2} & & & & 2\frac{EJ_z}{L^2} & & -6\frac{EJ_z}{L^2} & & & & 4\frac{EJ_z}{L^2}
\end{array}\right]
\begin{array}{c}
F_{ix} \\ F_{iy} \\ F_{iz} \\ M_{ix} \\ M_{iy} \\ M_{iz} \\ F_{jx} \\ F_{jy} \\ F_{jz} \\ M_{jx} \\ M_{jy} \\ M_{jz}
\end{array}
\end{array}
\qquad (255)
$$

3.9. Tranformation of the stiffness matrix to the global coordinate system

The element stiffness matrix should be transformed to the global system. The transformation method of the matrix of a frame element is analogous to the transformation of an element of a 3D truss presented in Chapter 3 but the third rotation around the x axis of the local system is necessary in order to lead axes y and z to the position of the principal central axes of inertia of an element cross section. Such a choice of local axes is very important for building the stiffness matrix which has been noted at the beginning of this chapter. The location of an element in space, applied types of coordinate systems and rotation angles notations are presented inFigure 42.

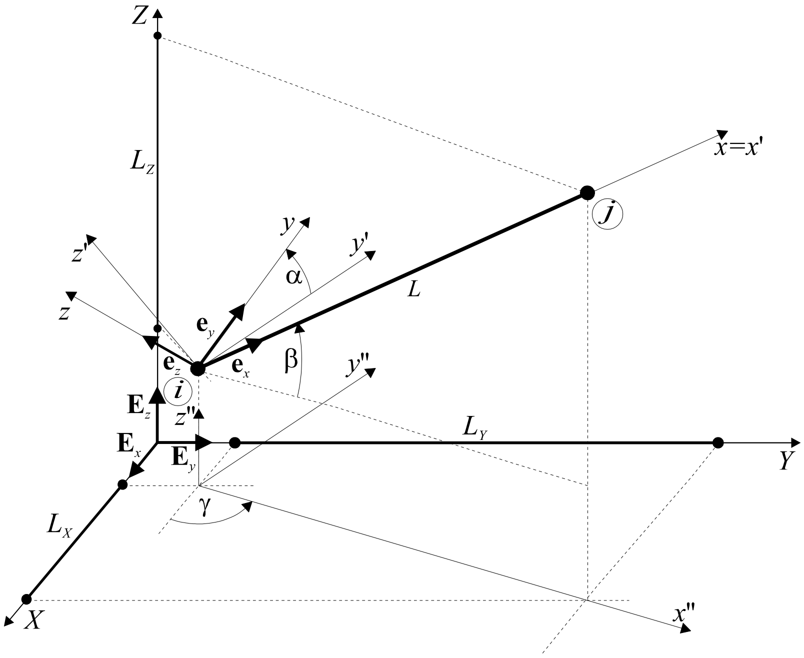


Figure 42. The frame element arrangement with regard to the global coordinate system and the notation of basic vectors.

In Figure 42the following notations are used: $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ as basic vectors of axes of the local coordinate system and $\mathbf{E}_X$, $\mathbf{E}_Y$, $\mathbf{E}_Z$ as basic vectors of axes of the global coordinate system. They will be helpful in subsequent transformations.

3.9.1. Use of the rotation angle α for building the transformation matrix

Now we perform the transformation of a certain displacement vector $\mathbf{u}'_i$ from the local system to the global one by the composition of three rotations:

$$\mathbf{u}_i = \mathbf{R}_\gamma \left[\mathbf{R}_\beta \left(\mathbf{R}_\alpha \mathbf{u}'_i \right) \right], \tag{256}$$

where $$\mathbf{R}_\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\alpha & -s_\alpha \\ 0 & s_\alpha & c_\alpha \end{bmatrix}, \tag{257}$$

is the rotation matrix around the x axis by an angle α,

$$\mathbf{R}_\beta = \begin{bmatrix} c_\beta & 0 & -s_\beta \\ 0 & 1 & 0 \\ s_\beta & 0 & c_\beta \end{bmatrix}, \tag{258}$$

is the rotation matrix around the y' axis by an angle β,

$$\mathbf{R}_\gamma = \begin{bmatrix} c_\gamma & -s_\gamma & 0 \\ s_\gamma & c_\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{259}$$

is the rotation matrix around the z'' axis by an angle γ. In Eqn. (257), (258)and (259)we have $c_\alpha = \cos\alpha$, $s_\alpha = \sin\alpha$, $c_\beta = \cos\beta$, $s_\beta = \sin\beta$, $c_\gamma = \cos\gamma$ and $s_\gamma = \sin\gamma$. Eqn. (256) can be written in a simpler way:

$$\mathbf{u}_i = \mathbf{R}_i \mathbf{u}'_i, \tag{260}$$

where $\mathbf{R}_i = \mathbf{R}_\gamma \mathbf{R}_\beta \mathbf{R}_\alpha$ is the transformation matrix and the inverse relation is:

$$\mathbf{u}'_i = (\mathbf{R}_i)^{\mathrm{T}} \mathbf{u}_i, \tag{261}$$

where $(\mathbf{R}_i)^{\mathrm{T}} = (\mathbf{R}_\alpha)^{\mathrm{T}} (\mathbf{R}_\beta)^{\mathrm{T}} (\mathbf{R}_\gamma)^{\mathrm{T}}$.

With this method of transformation, functions of angles γ and β can be determined on the basis of nodal coordinates of an element (they depend on direction cosines of an element - comp. Sec.3.2) and the angle α is an additional parameter which has to be given for all elements.

3.9.2. Use of a direction vector

Here we will present another way of determining the transformation matrix. Let an additional parameter determining an element be a direction vector $\mathbf{e}_y$ (Figure 42) which is located on the y axis of the local system and its modulus is equal to unity (such a vector is called a basic vector or a *versor* of an axis). Hence we have:

- vector of the x axis of the local system determined on the basis of element coordinates (its components are direction cosines of the element)

$$\mathbf{e}_x = \begin{bmatrix} e_{xX} \\ e_{xY} \\ e_{xZ} \end{bmatrix} = \frac{1}{L} \begin{bmatrix} L_X \\ L_Y \\ L_Z \end{bmatrix}, \tag{262}$$

- given direction vector of the element

$$\mathbf{e}_y = \begin{bmatrix} e_{yX} \\ e_{yY} \\ e_{yZ} \end{bmatrix}. \tag{263}$$

We look for the third basic vector $\mathbf{e}_z$ which allows us to write the transformation of any vector from the local coordinate system *xyz* to the global one *XYZ*.

Since the system *xyz* is the right cartesian coordinate system, then the versors of this system are orthogonal. Thus, we can write

$$\mathbf{e}_z = \mathbf{e}_x \times \mathbf{e}_y, \tag{264}$$

and from here we calculate

$$\mathbf{e}_z = \begin{bmatrix} e_{zX} \\ e_{zY} \\ e_{zZ} \end{bmatrix}, \tag{265}$$

where

$$e_{zX} = \begin{vmatrix} e_{xY} & e_{xZ} \\ e_{yY} & e_{yZ} \end{vmatrix}, \quad e_{zY} = -\begin{vmatrix} e_{xX} & e_{xZ} \\ e_{yX} & e_{yZ} \end{vmatrix}, \quad e_{zZ} = \begin{vmatrix} e_{xX} & e_{xY} \\ e_{yX} & e_{yY} \end{vmatrix}, \tag{266}$$

are the coordinates of the versor of the local *z* axis with regard to the global coordinate system.

Since any vector can be presented as a sum of products of its coordinates and versors, then we obtain:

$$\begin{aligned} \mathbf{u} &= u_x\mathbf{e}_x + u_y\mathbf{e}_y + u_z\mathbf{e}_z = u_x\left(e_{xX}\mathbf{E}_X + e_{xY}\mathbf{E}_Y + e_{xZ}\mathbf{E}_Z\right) + \\ &+ u_y\left(e_{yX}\mathbf{E}_X + e_{yY}\mathbf{E}_Y + e_{yZ}\mathbf{E}_Z\right) + u_z\left(e_{zX}\mathbf{E}_X + e_{zY}\mathbf{E}_Y + e_{zZ}\mathbf{E}_Z\right) = \\ &= \left(u_x e_{xX} + u_y e_{yX} + u_z e_{zX}\right)\mathbf{E}_X + \left(u_x e_{xY} + u_y e_{yY} + u_z e_{zY}\right)\mathbf{E}_Y + \\ &+ \left(u_x e_{xZ} + u_y e_{yZ} + u_z e_{zZ}\right)\mathbf{E}_Z, \end{aligned} \tag{267}$$

or less

$$\mathbf{u}_i = \mathbf{R}_i\mathbf{u}'_i, \tag{268}$$

where $\mathbf{R}_i$ is the rotation matrix of a node

$$\mathbf{R}_i = \begin{bmatrix} e_{xX} & e_{yX} & e_{zX} \\ e_{xY} & e_{yY} & e_{zY} \\ e_{xZ} & e_{yZ} & e_{zZ} \end{bmatrix}. \tag{269}$$

3.9.3. Use of a direction point

The necessity to give the direction vector in the form Eqn. (263)often causes difficulties during data input. Here we present one of the possibilities of simplifying the way of passing the direction of an element axis which is used in the *Autodesk Simulation Mechanical (ALGOR)* system. The 3D frame element is determined by three points (*i* - the first node, *j* - the last node, *k* - the direction node). The points *i*, *j*, *k* determine a plane in the three dimensional space. The axis *y* of the local coordinate system is in this plane. The *x* axis is determined by the line passing through points *i*, *j*. We find coordinates of versors for such a definition of directions of the local axes. Let X_i, Y_i, Z_i denote coordinates of the point *i* in the global system. If analogy, we denote coordinates of points *j* and *k*, then the element coordinates in the global system are equal to

$$L_X = X_j - X_i\,,\;\; L_Y = Y_j - Y_i\,,\;\; L_Z = Z_j - Z_i\,,\;\; L = \sqrt{L_X^2 + L_Y^2 + L_Z^2}\,, \tag{270}$$

and from here we calculate the components of vector $\mathbf{e}_x$:

$$e_{xX} = \frac{L_X}{L}\,,\;\; e_{xY} = \frac{L_Y}{L}\,,\;\; e_{xZ} = \frac{L_Z}{L}\,. \tag{271}$$

We form the vector $\mathbf{v}$ connecting the point *i* and the direction point *k* (Figure 43):

$$\mathbf{v} = \begin{bmatrix} X_k - X_i \\ Y_k - Y_i \\ Z_k - Z_i \end{bmatrix}. \tag{272}$$

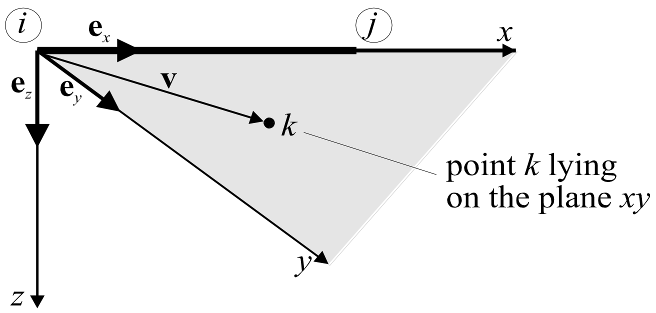


Figure 43. The direction point application to determine the y axis of the local coordinate system.

The vector product of the vectors $\mathbf{e}_x$ and $\mathbf{v}$ give a vector which is perpendicular to the xy plane. This vector will be the versor $\mathbf{e}_z$:

$$\mathbf{w} = \mathbf{e}_x \times \mathbf{v}, \tag{273}$$

$$w_X = \begin{vmatrix} e_{xY} & e_{xZ} \\ v_Y & v_Z \end{vmatrix}, \quad w_Y = -\begin{vmatrix} e_{xX} & e_{xZ} \\ v_X & v_Z \end{vmatrix}, \quad w_Z = \begin{vmatrix} e_{xX} & e_{xY} \\ v_X & v_Y \end{vmatrix}$$

$$w = \sqrt{w_X^2 + w_Y^2 + w_Z^2}, \tag{274}$$

$$\mathbf{e}_{zX} = \frac{w_X}{w}, \quad \mathbf{e}_{zY} = \frac{w_Y}{w}, \quad \mathbf{e}_{zZ} = \frac{w_Z}{w}. \tag{275}$$

Now we obtain the coordinates of the versor $\mathbf{e}_y$ from the vector product of the versor $\mathbf{e}_z$ by $\mathbf{e}_x$:

$$\mathbf{e}_y = \mathbf{e}_z \times \mathbf{e}_x, \tag{276}$$

$$e_{yX} = \begin{vmatrix} e_{zY} & e_{zZ} \\ e_{xY} & e_{xZ} \end{vmatrix}, \quad e_{yY} = -\begin{vmatrix} e_{zX} & e_{zZ} \\ e_{xX} & e_{xZ} \end{vmatrix}, \quad e_{yZ} = \begin{vmatrix} e_{zX} & e_{zY} \\ e_{xX} & e_{xY} \end{vmatrix}. \tag{277}$$

On the basis of results Eqn. (271), (275)and (277) we can form the transformation matrix $\mathbf{R}_i$ as in Eqn. (269).

3.9.4. The transformation matrix of an element

Now we build the transformation matrix of an element. Nodal displacement vectors and nodal force vectors have been grouped so that we can divide them into blocks containing either displacements or rotations and either forces or moments respectively. After this operation we can transform every block independently

$$\mathbf{R}^e = \begin{bmatrix} \mathbf{R}_i & & & \\ & \mathbf{R}_i & & \\ & & \mathbf{R}_j & \\ & & & \mathbf{R}_j \end{bmatrix}, \tag{278}$$

where $\mathbf{R}_i$ is the rotation matrix of the node i and $\mathbf{R}_j$ is the rotation matrix of the node j. Since the element is straight, as it was in previous cases (2D and 3D trusses, 2D frame), we assume $\mathbf{R}_i = \mathbf{R}_j$.

We obtain the transformation of the stiffness matrix to the global system by multiplying matrices identically as in Eqn. (95).

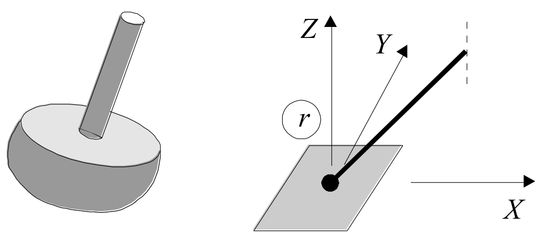
$$\mathbf{K}^e = \mathbf{R}^e \mathbf{K}'^e \left(\mathbf{R}^e\right)^{\mathrm{T}}, \tag{279}$$

where $\mathbf{R}^e$ is determined by Eqn. (278). The form of the matrix $\mathbf{K}^e$ is too complex in the global system, so we will not give it.

3.10. Boundary conditions for a 3D frame

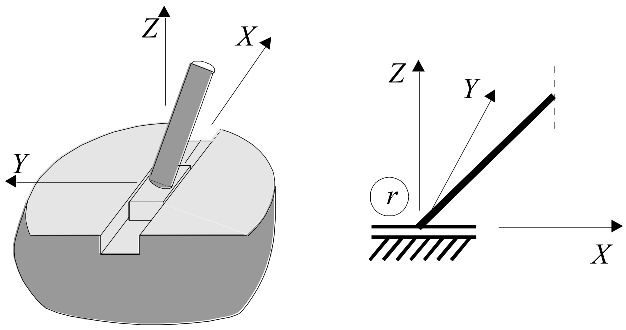
Boundary conditions existing in 3D frame supports are very similar to conditions described for two-dimensional frames. Differences concerning degrees of freedom which do not exist in plane frames are obvious. We elaborate only those boundary conditions which describe frame supports of space structures (Figure 44) and which are most often applied.

a) rigid support



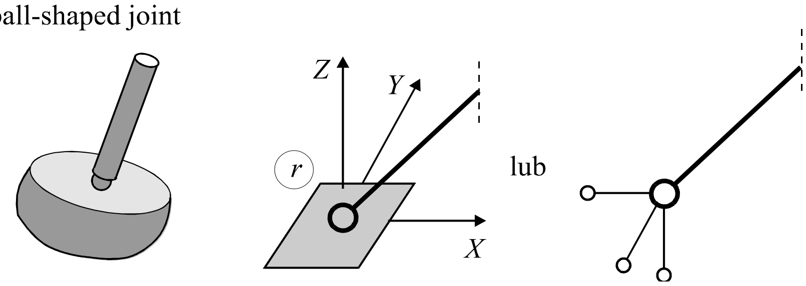
$u_{rX} = 0 \quad \varphi_{rX} = 0$
$u_{rY} = 0 \quad \varphi_{rY} = 0$
$u_{rZ} = 0 \quad \varphi_{rZ} = 0$

b) linear moveable support (along the X axis)



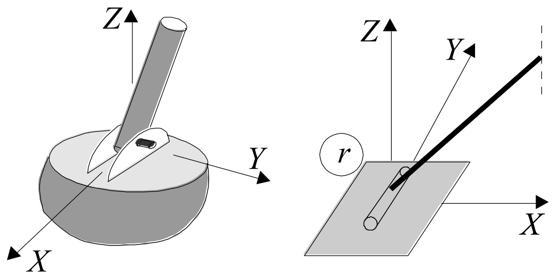
$u_{rY} = 0 \quad \varphi_{rX} = 0$
$u_{rZ} = 0 \quad \varphi_{rY} = 0$
$\varphi_{rZ} = 0$

c) ball-shaped joint



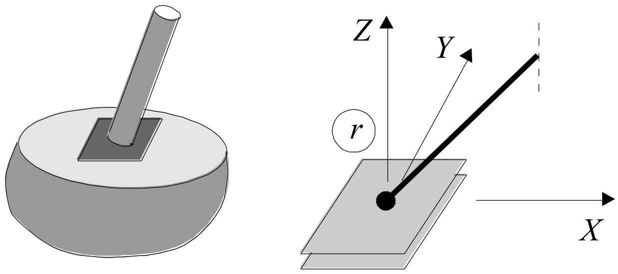
$u_{rX} = 0$
$u_{rY} = 0$
$u_{rZ} = 0$

d) cylindrical joint (rotation around the *Y* axis)



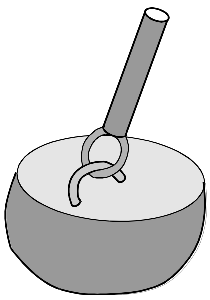
$u_{rX} = 0$ $\varphi_{rX} = 0$
$u_{rY} = 0$ $\varphi_{rZ} = 0$
$u_{rZ} = 0$

e) moveable plane support (a displacement on the *XY* plane)



$u_{rZ} = 0$

f) Cardan joint



$u_{rX} = 0$ $\varphi_{rZ} = 0$
$u_{rY} = 0$
$u_{rZ} = 0$

Figure 44. 3D frame support types.

Modification of the global stiffness matrix (comp. point 2.6) is the way of considering boundary conditions just as we have done in reference to previously described structures.

3.11. Boundary elements

A choice of supports to be used in a space structure increases if we add elastic constraints and ‘skew’ supports.

As in previous chapters, we propose to use elastic and fixed boundary elements for modelling these constraints. In fact we can use a single element described in Chapters 2 or 3 of which we can compose a more complex support but for convenience we will show here the use of the matrix of a versatile elastic element with six degrees of freedom:

$$\mathbf{K}'^{b} = \begin{bmatrix} h_{rX} & 0 & 0 & 0 & 0 & 0 \\ 0 & h_{rY} & 0 & 0 & 0 & 0 \\ 0 & 0 & h_{rZ} & 0 & 0 & 0 \\ 0 & 0 & 0 & g_{rX} & 0 & 0 \\ 0 & 0 & 0 & 0 & g_{rY} & 0 \\ 0 & 0 & 0 & 0 & 0 & g_{rZ} \end{bmatrix}, \tag{280}$$

where h_{rX}, h_{rY}, h_{rZ} are spring rates and g_{rX}, g_{rY}, g_{rZ} are flexural (or torsion) stiffness of springs.

The transformation of this matrix to the global system is similar to the one presented in Chapter 4 (Eqn. (212)). Since reactions of our elements are contained in two independent vectors: the vector of support forces and the vector of support moments, then the transformation matrix has the form:

$$\mathbf{R}^{b} = \begin{bmatrix} \mathbf{R}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_r \end{bmatrix}, \tag{281}$$

where $\mathbf{R}_r$ is the rotation matrix of the node given by Eqn. (269). After the multiplication we obtain the stiffness matrix of the boundary element in the global coordinate system:

$$\mathbf{K}^{b} = \mathbf{R}^{b}\mathbf{K}'^{b}\left(\mathbf{R}^{b}\right)^{\mathsf{T}} = \begin{bmatrix} \mathbf{H} & \mathbf{0} \\ \mathbf{0} & \mathbf{G} \end{bmatrix}, \tag{282}$$

where **H** is the stiffness matrix for a movement and **G** is the stiffness matrix for a rotation:

$$\mathbf{H} = \begin{bmatrix} e_{xX}^2 h_{rX} + e_{yX}^2 h_{rY} + e_{zX}^2 h_{rZ} & 0 & 0 \\ 0 & e_{xY}^2 h_{rX} + e_{yY}^2 h_{rY} + e_{zY}^2 h_{rZ} & 0 \\ 0 & 0 & e_{xZ}^2 h_{rX} + e_{yZ}^2 h_{rY} + e_{zZ}^2 h_{rZ} \end{bmatrix}. \tag{283}$$

It is easy to obtain the matrix **G** from the matrix **H** changing the stiffness of tension of springs h_{rX}, h_{rY}, h_{rZ} into the stiffness of bending springs g_{rX}, g_{rY}, g_{rZ}.

4. Two-dimensional elements

Structures discussed in the previous chapters were modelled by means of bar structures whose equilibrium equations as well as their geometrical relationships are described with the help of differential equilibrium equations and whose independent variable is measured along the bar axis. This rather simple structure lets us get familiar with the essence of the FEM and convinces the reader that this method is efficient in solving very complex and extended problems in structural mechanics. Now, we will discuss surface structures such as 2D elements, plate and shell for which displacements, strains, internal forces are the functions of two independent coordinates. As a result, equilibrium equations are partial differential equations much more difficult to be solved than ordinary equations.

Differential equilibrium equations for bar structures are simple enough to be integrated. Their exact results can be used as element shape functions. The situation is quite different for surface structures. Partial differential equations describing the equilibrium of those structures have unique solutions only for very simple problems. Solutions obtained by using the approximation method (for example, by expansion in a series) are very laborious and they require a lot of work and therefore a computer has to be used in order to solve a set of equations and sum series. In such a situation, a numerical method which assumes some simplification at the stage of formation of element equilibrium equations appears to be more effective. That is why the finite element method has brought so many significant results to continuum mechanics. It can be easily noticed in the example of a two-dimensional element which is the case of the simplest continuum. The 2D element (slab element) can be defined as a solid of which one dimension (thickness) is considerably smaller than the two others and whose middle plane (the surface parallel to both external surfaces of an element) is a plane (Figure 45). A plate element has also such a shape but the slab element differs from a plate the way it is loaded. The slab element can be loaded only with the load acting in its plane and by the temperature dependent upon the x and y coordinates. On the other hand, the plate can be loaded with a force perpendicular to its surface or any temperature field. Plate elements will be discussed in the following chapter.

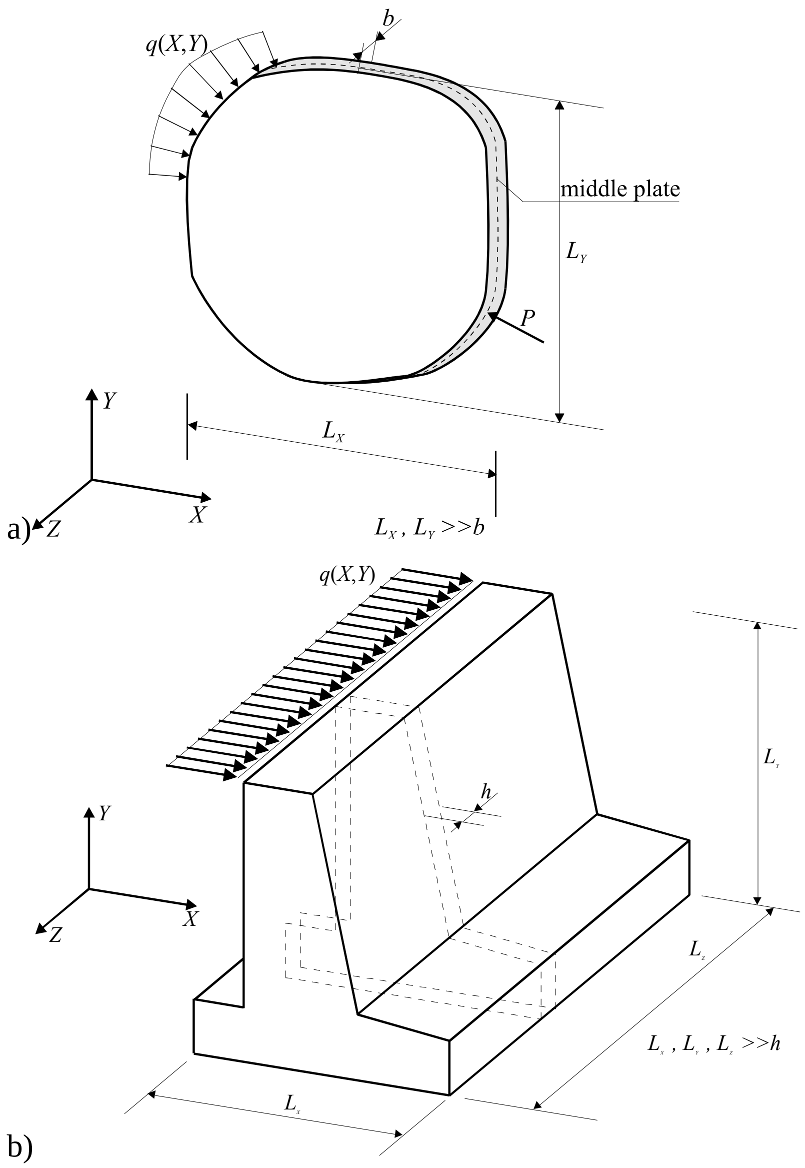


Figure 45. The exemplary application of a slab – 2D element.

4.1. Plane stress and strain

When external surfaces of a 2D element are free and this element is thin enough, we can assume that $\sigma_z = 0, \tau_{zx} = 0, \tau_{zy} = 0$ in reference to the whole thickness of the element. Then it is said that this is a plane stress problem. The thinner the 2D element (comp. Nowacki (1979), Timoshenko and Goodier (1962)), the better the approximation is. Hence only the components of stress shown in Figure 46 are non-zero.

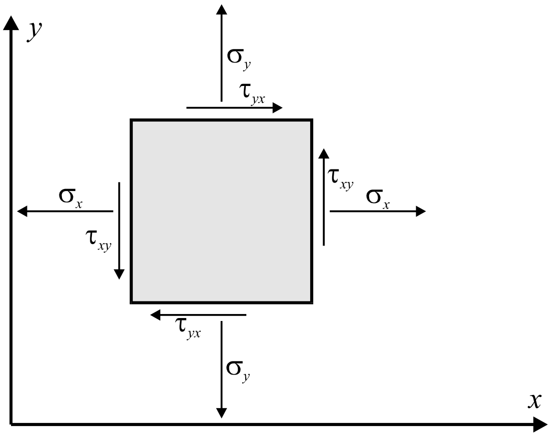


Figure 46. Stress tensor components in plane stress.

With regard to the symmetry of a stress tensor components of shear stress τ_{xy} and τ_{yx} are equal, thus we have three independent components of stress which we compose in the stress vector:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}. \tag{284}$$

A completely different case occurs when the component $L_Z$in Figure 45b is very significant, that is $h<<L_X$, L_Y, L_Z, and the support and load conditions are constant along the axis which is perpendicular to the element. The structure satisfying these conditions can also be analysed by applying plane state which in fact is plane strain. Since the cross dimension of the structure shown in Figure 45b prevents the structure deformation in the direction perpendicular to the cross section, the thin layer cut out from this structure is in the state described by the equation:

$$\varepsilon_z = 0, \quad \gamma_{zx} = 0, \quad \gamma_{zy} = 0. \tag{285}$$

$\sigma_z \neq 0$ comes from the above equations, but the first equation allows to calculate the component σ_z on the basis of two other components of a direct stress. Thus, we have

$$\sigma_z = \nu\left(\sigma_x + \sigma_y\right), \tag{286}$$

which allows to limit the number of searched components of the stress vector to three components given in Eqn. (284).

We also group independent components of the strain tensor in a column matrix which we have called a strain vector:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}. \tag{287}$$

There is a relationship between vectors **σ** and **ε** described by constitutive equations whose form depends on the model of the material which the structure is made of. In this book we deal only with elastic isotropic materials which obey Hook's law. Hence we can write the constitutive equation as follows:

$$\boldsymbol{\sigma} = \mathbf{D}\cdot\boldsymbol{\varepsilon}, \tag{288}$$

where **D** is a square matrix containing material elastic constants described in Chapter 1.

For plane stress, the matrix **D**has the form written by Eqn. (13). Plane strain requires another matrix for elastic constants which is described by Eqn. (17).

4.2. Geometric relationships

A certain point can move only on the plane during the deformation process and then the displacement vector of this point $\mathbf{u}(x,y)$ has two components:

$$\mathbf{u}(x,y) = \begin{bmatrix} u_x(x,y) \\ u_y(x,y) \end{bmatrix}. \tag{289}$$

Some known relations exist (Timoschenko and Goodier (1962)) between the components of displacement and strain vectors:

$$\varepsilon_x = \frac{\partial u_x}{\partial x}, \quad \varepsilon_y = \frac{\partial u_y}{\partial y}, \quad \gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}, \tag{290}$$

which can be presented in the form:

$$\boldsymbol{\varepsilon} = \mathcal{D}\,\mathbf{u}(x,y), \tag{291}$$

where $\mathcal{D}$ is the matrix of differential operators Eqn. (35).

4.3. The stiffness matrix of an elastic element

Let us divide a continuum into finite elements. We will discuss only a triangular 2D element in this book and we will choose such elements during discretization (Figure 47).

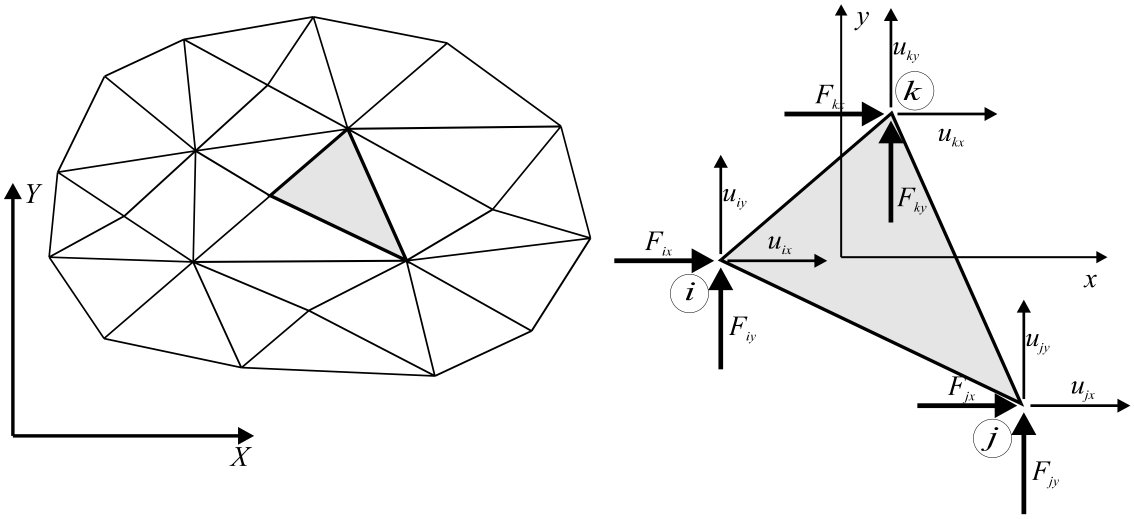


Figure 47. Nodal forces and displacements for the 2D element in the global coordinate system.

According to assumption Eqn. (289)it is seen that every node of an element has two degrees of freedom and all nodal forces have two components. The local coordinate system *xy* is chosen in such a way that its axes are parallel to the axes of the global coordinate system. Hence distinguishing components of local and global vectors and matrices is insignificant.

Now we group nodal displacements and forces in the vectors of:

- nodal and element displacements

$$\mathbf{u}_i = \begin{bmatrix} u_{ix} \\ u_{iy} \end{bmatrix},\ \mathbf{u}_j = \begin{bmatrix} u_{jx} \\ u_{jy} \end{bmatrix},\ \mathbf{u}_k = \begin{bmatrix} u_{kx} \\ u_{ky} \end{bmatrix},\ \mathbf{u}^e = \begin{bmatrix} \mathbf{u}_i \\ \mathbf{u}_j \\ \mathbf{u}_k \end{bmatrix} = \begin{bmatrix} u_{ix} \\ u_{iy} \\ u_{jx} \\ u_{jy} \\ u_{kx} \\ u_{ky} \end{bmatrix} \tag{292}$$

- nodal and element forces

$$\mathbf{f}_i = \begin{bmatrix} F_{ix} \\ F_{iy} \end{bmatrix},\ \mathbf{f}_j = \begin{bmatrix} F_{jx} \\ F_{jy} \end{bmatrix},\ \mathbf{f}_k = \begin{bmatrix} F_{kx} \\ F_{ky} \end{bmatrix},\ \mathbf{f}^e = \begin{bmatrix} \mathbf{f}_i \\ \mathbf{f}_j \\ \mathbf{f}_k \end{bmatrix} = \begin{bmatrix} F_{ix} \\ F_{iy} \\ F_{jx} \\ F_{jy} \\ F_{kx} \\ F_{ky} \end{bmatrix}. \tag{293}$$

Since we look for the dependence between nodal displacement and nodal forces vectors of an element we apply the principle of virtual work (comp. Chapter 1) which requires giving the relation between displacements of points lying within the element and displacements of nodes. Accepting errors coming from approximation, we assume that this relationship can be written by the function of two variables:

$$u_x(x,y) = N_i(x,y)u_{ix} + N_j(x,y)u_{jx} + N_k(x,y)u_{kx} \quad \text{and}$$
$$u_y(x,y) = N_i(x,y)u_{iy} + N_j(x,y)u_{jy} + N_k(x,y)u_{ky}, \tag{294}$$

or the general matrix form:

$$\mathbf{u}(x,y) = \mathbf{N}^e(x,y)\mathbf{u}^e, \tag{295}$$

where $\mathbf{N}^e(x,y)$ is the matrix of shape functions of the element:

$$\mathbf{N}^e(x,y) = \left[N_i(x,y)\ \mathbf{I} \quad N_j(x,y)\ \mathbf{I} \quad N_k(x,y)\ \mathbf{I}\right], \tag{296}$$

and $N_i(x,y)$, $N_j(x,y)$, $N_k(x,y)$ are the shape functions for nodes i, j, k.

Let us now assume the simplest of all possible forms of the shape function for the node i

$$N_i(x,y) = a_i + b_i x + c_i y, \tag{297}$$

where a_i, b_i, c_i are constants which we determine on the basis of consistency conditions

$$N_i(x_i, y_i) = 1,\ N_i(x_j, y_j) = 0,\ N_i(x_k, y_k) = 0. \tag{298}$$

After inserting these conditions into Eqn. (297), we obtain the set of equations:

$$\begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix} \begin{bmatrix} a_i \\ b_i \\ c_i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{299}$$

which, after being solved, give the values of coefficients of the shape function.

Eqn. (299) can also be written in the general form:

$$\mathbf{M}\boldsymbol{\alpha}_i = \boldsymbol{\delta}_i \text{, where } \boldsymbol{\delta}_i = \begin{bmatrix} \delta_{i1} \\ \delta_{i2} \\ \delta_{i3} \end{bmatrix} \tag{300}$$

which, after modification depending on the change of i into j (or k), allows us to determine the coefficients of the shape functions for the subsequent nodes. δ_{ij} means the Kronecker's delta in this equation.

We solve the set of Eqn. (299) by the Cramer method

$$W = \det \mathbf{M} = \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix} = \begin{vmatrix} x_j & y_j \\ x_k & y_k \end{vmatrix} - \begin{vmatrix} x_i & y_i \\ x_k & y_k \end{vmatrix} + \begin{vmatrix} x_i & y_i \\ x_j & y_j \end{vmatrix},$$

$$W_{a_i} = \begin{vmatrix} 1 & x_i & y_i \\ 0 & x_j & y_j \\ 0 & x_k & y_k \end{vmatrix} = \begin{vmatrix} x_j & y_j \\ x_k & y_k \end{vmatrix},$$

$$W_{b_i} = \begin{vmatrix} 1 & 1 & y_i \\ 1 & 0 & y_j \\ 1 & 0 & y_k \end{vmatrix} = -\begin{vmatrix} 1 & y_i \\ 1 & y_k \end{vmatrix} = y_j - y_k, \tag{301}$$

$$W_{c_i} = \begin{vmatrix} 1 & x_i & 1 \\ 1 & x_j & 0 \\ 1 & x_k & 0 \end{vmatrix} = \begin{vmatrix} 1 & y_j \\ 1 & y_k \end{vmatrix} = x_k - x_j$$

then $a_i = \dfrac{W_{a_i}}{W}$, $b_i = \dfrac{W_{b_i}}{W}$, $c_i = \dfrac{W_{c_i}}{W}$.

Similarly, if we change the index i into j and we find $\boldsymbol{\delta}_j = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$,

$$W_{a_j} = \begin{vmatrix} 0 & x_i & y_i \\ 1 & x_j & y_j \\ 0 & x_k & y_k \end{vmatrix} = -\begin{vmatrix} x_i & y_i \\ x_k & y_k \end{vmatrix},$$

$$W_{b_j} = \begin{vmatrix} 1 & 0 & y_i \\ 1 & 1 & y_j \\ 1 & 0 & y_k \end{vmatrix} = y_k - y_i, \tag{302}$$

$$W_{c_j} = \begin{vmatrix} 1 & x_i & 0 \\ 1 & x_j & 1 \\ 1 & x_k & 0 \end{vmatrix} = x_i - x_k,$$

$$a_j = \frac{W_{a_j}}{W},\ b_j = \frac{W_{b_j}}{W},\ c_j = \frac{W_{c_j}}{W}.$$

Finally, we have

$$\boldsymbol{\delta}_k = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$W_{a_k} = \begin{vmatrix} 0 & x_i & y_i \\ 0 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix} = \begin{vmatrix} x_i & y_i \\ x_j & y_j \end{vmatrix},$$

$$W_{b_k} = \begin{vmatrix} 1 & 0 & y_i \\ 1 & 0 & y_j \\ 1 & 1 & y_k \end{vmatrix} = y_i - y_j\,, \tag{303}$$

$$W_{c_k} = \begin{vmatrix} 1 & x_i & 0 \\ 1 & x_j & 0 \\ 1 & x_k & 1 \end{vmatrix} = x_j - x_i\,,$$

$$a_k = \frac{W_{a_k}}{W},\ b_k = \frac{W_{b_k}}{W},\ c_k = \frac{W_{c_k}}{W}\ .$$

for node k.

Constants a_i, a_j, a_k are insignificant for further transformations (because they are connected with the rigid motion of a 2D element) and they can be neglected when solving the set of Eqn. (300).

After determining the shape functions of the element, let us come back to its strains. We insert Eqn. (295)in (291):

$$\boldsymbol{\varepsilon} = \mathscr{D}\mathbf{N}^e(x,y)\,\mathbf{u}^e = \mathbf{B}^e(x,y)\,\mathbf{u}^e\,, \tag{304}$$

obtaining the dependence between the nodal displacements of the element and its strains. The matrix **B** in Eqn. (304) is called a geometric matrix and it can be expressed as follows:

$$\mathbf{B}^e(x,y) = \begin{bmatrix} \mathbf{B}_i(x,y) & \mathbf{B}_j(x,y) & \mathbf{B}_k(x,y) \end{bmatrix},$$

where $\mathbf{B}_n = \mathscr{D}\mathbf{N}_n(x,y) = \begin{bmatrix} b_n & 0 \\ 0 & c_n \\ c_n & b_n \end{bmatrix}$ (305)

is the geometric matrix of any node n.

Thus, we have all components which are necessary to write an element equilibrium equation. We apply the principle of virtual work which says that the external work (done by external forces - here nodal forces) has to be equal to internal work (done by stress) of a 2D element:

$$\left(\mathbf{u}^e\right)^{\mathsf{T}}\mathbf{f}^e = \int_{\mathcal{V}} \boldsymbol{\varepsilon}^{\mathsf{T}}\boldsymbol{\sigma}\, d\mathcal{V}\,. \tag{306}$$

We transform this equation first substituting the constitutive relation Eqn. (288) for **δ** and next substituting geometric relations (304)for **ε**:

$$\left(\mathbf{u}^e\right)^{\mathsf{T}}\mathbf{f}^e = \int_{\mathcal{V}} \left(\mathbf{B}^e\mathbf{u}^e\right)^{\mathsf{T}}\mathbf{D}\mathbf{B}^e\mathbf{u}^e d\mathcal{V} = \left(\mathbf{u}^e\right)^{\mathsf{T}}\int_{\mathcal{V}}\left(\mathbf{B}^e\right)^{\mathsf{T}}\mathbf{D}\mathbf{B}^e d\mathcal{V}\mathbf{u}^e\,. \tag{307}$$

In this equation the nodal displacement vectors of the element being independent of variables x and y, are taken to the front and back of the integral. Eqn. (307) can be solved independently of element displacements only when

$$\mathbf{f}^e = \int_{\mathcal{V}}\left(\mathbf{B}^e\right)^{\mathsf{T}}\mathbf{D}\mathbf{B}^e d\mathcal{V}\ \mathbf{u}^e\,, \tag{308}$$

which, after comparison with the known relation, was referred to in all previous chapters of this book:

$$\mathbf{f}^e = \mathbf{K}^e\,\mathbf{u}^e\,,$$

gives us the equation determining coefficients of the element stiffness matrix:

$$\mathbf{K}^e = \int_{\mathcal{V}}\left(\mathbf{B}^e\right)^{\mathsf{T}}\mathbf{D}\mathbf{B}^e d\mathcal{V}\,. \tag{309}$$

Building the element stiffness matrix can be considerably easy if we note that this matrix divides into blocks:

$$\mathbf{K}^e = \begin{bmatrix} \mathbf{K}_{ii} & \mathbf{K}_{ij} & \mathbf{K}_{ik} \\ \mathbf{K}_{ji} & \mathbf{K}_{jj} & \mathbf{K}_{jk} \\ \mathbf{K}_{ki} & \mathbf{K}_{kj} & \mathbf{K}_{kk} \end{bmatrix}, \tag{310}$$

in which any of them, for example $\mathbf{K}_{ij}$, can be calculated from the equation:

$$\mathbf{K}_{ij} = \int_{\mathcal{V}}\left(\mathbf{B}_i\right)^{\mathsf{T}}\mathbf{D}\mathbf{B}_j\, d\mathcal{V}\,, \tag{311}$$

and others coming from analogous equations formed after suitable changes of indices have been made.

The insertion of the geometric matrices $\mathbf{B}_i$ and $\mathbf{B}_j$ given by Eqn. (305) and the matrix **D** given by Eqn. (13) into (311)results in

$$\mathbf{K}_{ij} = (\mathbf{B}_i)^{\mathrm{T}} \mathbf{D} \mathbf{B}_j \int_{\mathcal{V}} d\,\mathcal{V} = (\mathbf{B}_i)^{\mathrm{T}} \mathbf{D} \mathbf{B}_j A b =$$

$$= \frac{EAb}{1-\nu^2} \begin{bmatrix} b_i b_j + c_i c_j \frac{1-\nu}{2} & b_i c_j \nu + b_j c_i \frac{1-\nu}{2} \\ b_j c_i \nu + b_i c_j \frac{1-\nu}{2} & c_i c_j + b_i b_j \frac{1-\nu}{2} \end{bmatrix}, \tag{312}$$

where A is the surface of a slab element and b is the its thickness.

The above matrix is the stiffness matrix for plane stress.

Note that matrices $\mathbf{B}_i$, $\mathbf{B}_j$ and $\mathbf{D}$ do not contain components dependent on variables x, y, z, thus we can take them outside the integral.

We obtain the block of the stiffness matrix for plane strain accepting the matrix of material constants according to Eqn. (17):

$$\mathbf{K}_{ij} = \frac{EAb}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) b_i b_j + c_i c_j \frac{1-2\nu}{2} & b_i c_j \nu + b_j c_i \frac{1-2\nu}{2} \\ b_j c_i \nu + b_i c_j \frac{1-2\nu}{2} & (1-\nu) c_i c_j + b_i b_j \frac{1-2\nu}{2} \end{bmatrix}. \tag{313}$$

Since the local coordinate system is assumed in such a way that its axes are parallel to the global coordinate system, then we do not have to transform the stiffness matrix.

4.4. Element strain and stress

We also calculate element strains. They are given by Eqn. (304) and taking into consideration Eqn. (305) we have

$$\varepsilon_x = \sum_{n=i,j,k} b_n u_{nx}, \quad \varepsilon_y = \sum_{n=i,j,k} b_n u_{ny}, \quad \gamma_{xy} = \sum_{n=i,j,k} \left(c_n u_{nx} + b_n u_{ny} \right). \tag{314}$$

We see that components of the strain vector are constant within the element which is the consequence of the assumption of linear shape functions. This element is called CST (*constant strain triangle*).

We determine element stresses from the constitutive Eqn. (288) and Eqn. (13) or (17) according to the kind of variant that we deal with. It is obvious that strains, just as stresses are constant within the CST element.

4.5. A Nodal force vector for a distributed load

Loads on slab elements can be treated as loads on plane trusses which means that they can be applied to the nodes of a structure. But if a distributed load acting on the boundary of an element is given, then it should be converted to concentrated forces acting on the nodes of an element (Figure 48).

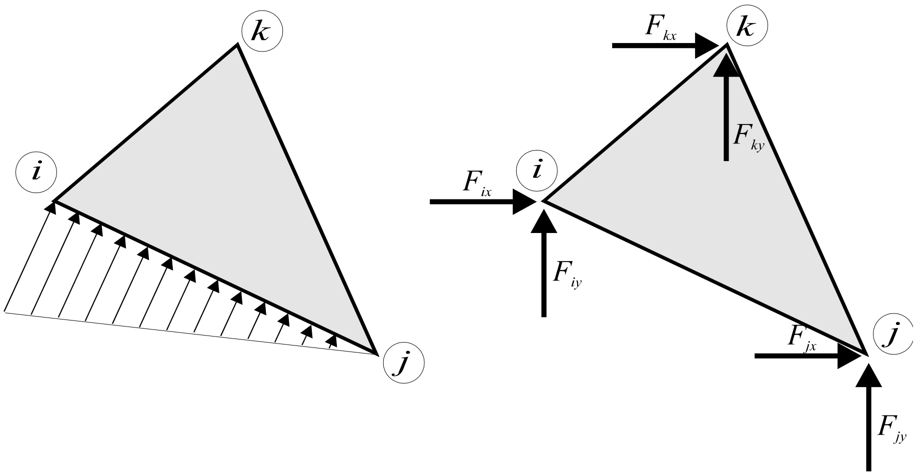


Figure 48. Nodal forces representing continuous loads.

Similarly, as in previous chapters, we apply the principal of virtual work giving the following equilibrium equation for this case:

$$\left(\mathbf{u}^e\right)^{\mathsf{T}}\mathbf{f}^e + L_{ij}\int_0^1 \mathbf{u}(\xi)^{\mathsf{T}}\mathbf{q}(\xi)d\xi = 0\,, \tag{315}$$

where $\mathbf{u}(\xi)$ contains functions describing the displacement of the loaded edge and $\mathbf{q}(\xi)=\begin{bmatrix} q_x(\xi) \\ q_y(\xi) \end{bmatrix}$ contains functions describing the load on the edge, L_{ij} is the length of the edge i-j, ξ is the non-dimensional coordinate taking zero value at the node i and value 1 at the node j. Since we assume linear shape functions for the element, then we write the vector $\mathbf{u}(\xi)$ as follows:

$$\mathbf{u}(\xi) = \mathbf{N}^e_{ij}\mathbf{u}^e\,, \tag{316}$$

where $\mathbf{N}^e_{ij}$ is the matrix of shape functions for displacements of the boundary.

$$\mathbf{N}^e_{ij} = \left[N^o_i(\xi)\,\mathbf{I} \quad N^o_j(\xi)\,\mathbf{I} \quad N^o_k(\xi)\,\mathbf{0}\right], \tag{317}$$

where $N^o_i(\xi) = 1-\xi$, $N^o_j(\xi) = \xi$,

or in the developed form

$$\mathbf{N}^e_{ij} = \begin{bmatrix} 1-\xi & 0 & \xi & 0 & 0 & 0 \\ 0 & 1-\xi & 0 & \xi & 0 & 0 \end{bmatrix}. \tag{318}$$

After inserting relation Eqn. (316)into Eqn. (315), we obtain

$$\mathbf{f}^{e} = -L_{ij}\int_{0}^{1}\left(\mathbf{N}_{ij}^{e}\right)^{\mathsf{T}}\mathbf{q}(\xi)d\xi\,, \tag{319}$$

After taking into consideration the shape functions described by Eqn. (318), we obtain

$$\mathbf{f}^{e} = -L_{ij}\int_{0}^{1}\begin{bmatrix}(1-\xi)q_{x}(\xi)\\(1-\xi)q_{y}(\xi)\\\xi q_{x}(\xi)\\\xi q_{y}(\xi)\\0\\0\end{bmatrix}d\xi\,. \tag{320}$$

For example, let us calculate the nodal force vector due to the linear distributed load on the edge *i-j* of value q_{ix}, q_{iy} - at the node *i* and q_{jx}, q_{jy} - at the node *j*. We write such a load with the help of a non-dimensional coordinate ξ:

$$\mathbf{q}(\xi) = \begin{bmatrix}q_{ix}(1-\xi)+q_{jx}\xi\\q_{iy}(1-\xi)+q_{jy}\xi\end{bmatrix}, \tag{321}$$

and after inserting the above equation into Eqn. (320), we obtain

$$\mathbf{f}^{e} = -L_{ij}\begin{bmatrix}q_{ix}\int_{0}^{1}(1-\xi)^{2}d\xi+q_{jx}\int_{0}^{1}(1-\xi)\xi d\xi\\q_{iy}\int_{0}^{1}(1-\xi)^{2}d\xi+q_{jy}\int_{0}^{1}(1-\xi)\xi d\xi\\q_{ix}\int_{0}^{1}(1-\xi)\xi d\xi+q_{jx}\int_{0}^{1}\xi^{2}d\xi\\q_{iy}\int_{0}^{1}(1-\xi)\xi d\xi+q_{jy}\int_{0}^{1}\xi^{2}d\xi\\0\\0\end{bmatrix}, \tag{322}$$

which after integration gives

$$\mathbf{f}^e = -\frac{L_{ij}}{6}\begin{bmatrix} 2q_{ix} + q_{jx} \\ 2q_{iy} + q_{jy} \\ q_{ix} + 2q_{jx} \\ q_{iy} + 2q_{jy} \\ 0 \\ 0 \end{bmatrix}. \tag{323}$$

For a particular case when the load is constant and equal to $\mathbf{q}(\xi) = \begin{bmatrix} q_{ox} \\ q_{oy} \end{bmatrix}$, on the basis of Eqn. (323) we obtain

$$\mathbf{f}^e = -\frac{L_{ij}}{2}\begin{bmatrix} q_{ox} \\ q_{oy} \\ q_{ox} \\ q_{oy} \\ 0 \\ 0 \end{bmatrix}. \tag{324}$$

It should be remembered that the calculated forces are forces acting on the element. We obtain the necessary nodal forces changing the sense of vectors which means:

$$\mathbf{p}^e = -\mathbf{f}^e , \tag{325}$$

where $\mathbf{p}^e$ is the nodal force vector for the nodes touching the element *e*.

4.6. A Nodal force vector due to a temperature load

As in the previous section, we apply the principal of virtual work to calculate alternative nodal forces replacing a temperature load. In accordance with the features of a CST element we will take into consideration only a constant temperature field within the element.

The suitable equation of virtual work has the form:

$$\left(\mathbf{u}^e\right)^{\mathsf{T}} \mathbf{f}^{et} = \int_{\mathcal{V}} \boldsymbol{\varepsilon}^{\mathsf{T}} \boldsymbol{\sigma}_t \, d\mathcal{V} = \int_{\mathcal{V}} \boldsymbol{\varepsilon}^{\mathsf{T}} \mathbf{D} \boldsymbol{\varepsilon}_t \, d\mathcal{V} , \tag{326}$$

where $\boldsymbol{\sigma}_t$ is the stress field in the element which is caused by the temperature and $\boldsymbol{\varepsilon}_t$ is the strain of the element caused by the change of a temperature.

Assuming isotropy of a 2D element we obtain

$$\boldsymbol{\varepsilon}_t = \alpha_t \Delta t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \tag{327}$$

After inserting geometric relation Eqn. (304) into Eqn. (326), we obtain

$$\mathbf{f}^{et} = \alpha_t \Delta t \int_{\mathcal{V}} \left(\mathbf{B}^e\right)^{\mathsf{T}} \mathbf{D} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} d\mathcal{V} = \alpha_t \Delta t A b \left(\mathbf{B}^e\right)^{\mathsf{T}} \mathbf{D} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}. \tag{328}$$

For a plane stress problem this equation is simplified to the following relation:

$$\mathbf{f}^{et}_{\mathrm{PSN}} == \frac{\alpha_t \Delta t E A b}{1-\nu} \begin{bmatrix} b_i \\ c_i \\ b_j \\ c_j \\ b_k \\ c_k \end{bmatrix}, \tag{329}$$

where b_i ... c_k are coefficients of shape functions of the CST element.

Plane strain gives a slightly different nodal force vector:

$$\mathbf{f}^{et}_{\mathrm{PSO}} == \frac{\alpha_t \Delta t E A b}{(1+\nu)(1-2\nu)} \begin{bmatrix} b_i \\ c_i \\ b_j \\ c_j \\ b_k \\ c_k \end{bmatrix}. \tag{330}$$

As in previous sections, we should change the signs of components of nodal forces before applying them to the nodes:

$$\mathbf{p}^{et} = -\mathbf{f}^{et}. \tag{331}$$

We calculate stresses in the element undergoing the action of a temperature taking into consideration strains caused by the thermal expansion of the element:

$$\boldsymbol{\sigma}_t = \mathbf{D}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_t) = \mathbf{D}\left(\mathbf{B}\mathbf{u}^e - \alpha_t \Delta t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right). \tag{332}$$

4.7. Boundary conditions of a 2D element

Boundary conditions of a two-dimensional structure can be treated analogously to the conditions in a plane truss because the nodes of both systems have two degrees of freedom on the *XY* plane.

Hence we have: fixed supports (at the node r_1 inFigure 49) and supports which can move along the *X* axis (at the node r_2), next supports which can move along the *Y* axis (at the node r_4) or skew supports (at the node r_3). The boundary conditions for these supports are as follows:

- node r_1: $u_{r_1X} = 0$, $u_{r_1Y} = 0$,
- node r_2: $u_{r_2Y} = 0$,
- node r_4: $u_{r_4X} = 0$,
- for noder_3, where constraints are not consistent with the axes of the global coordinate system we propose the use of boundary elements described in Chapter 2.

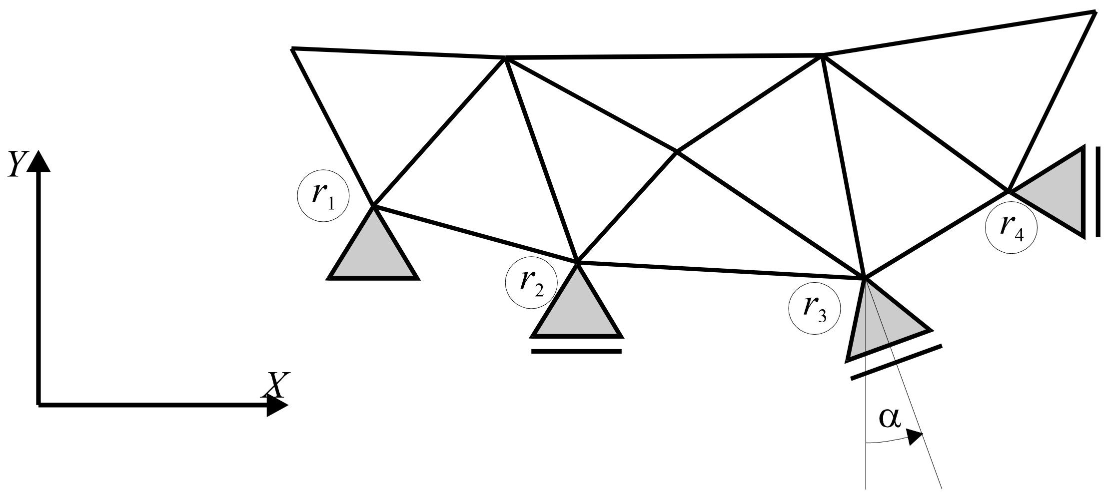


Figure 49. The 2D element (slab element) scheme divided into finite elements and supports.

5. Statics of plates

Plates are one of the most commonly used elements in structures. They can be found in almost every building or mechanical structure. The geometric shape of a plate can be defined similarly to a 2D element (Chapter 6), but they differ in the way of loading. Plates are loaded with normal loads to their surfaces which cause bending. Bending is not present in the case of the deformation of the 2D element.

Analytical methods of determining both deflections and internal forces were described by Euler, Bernoulli, Germain, Lagrange, Poisson and especially by Navier in papers which appeared at the end of the 18th century described by Rao (1982). Literature devoted to the theory of plates is unusually rich, the books of Kączkowski (1980), Nowacki (1979), Timoshenko and Woinowsky-Krieger (1962) are recommended to interested readers.

Many important statics and dynamics problems of plates were solved by analytical methods (mainly by the method of the Fourier series), but they are inaccurate both in the case of problems with complex boundary conditions and complicated shapes of plates. However, the finite element method has proved to be universal and although it gives approximate solutions, they are precise enough for practical applications.

5.1. Basic assumptions and equations of the classic theory of plates

We assume that these plates the assumptions of the classic theory of thin plates (Timoshenko and Woinowsky-Krieger (1962)):

a) thickness of a plate is small in comparison with its other dimensions;
b) deflections of plates are small in comparison with its thickness;
c) middle plane does not undergo lengthening (or shortening);
d) points lying on the lines which are perpendicular to the middle plane before its deformation lie on these lines after the deformation;
e) components of stress which are perpendicular to the plane of the plate can be neglected.

From point d) of the above assumptions it follows that the displacement of points lying within the plate varies linearly with its thickness (Figure 50):

$$u_x = -z\frac{\partial w}{\partial x}, \quad u_y = -z\frac{\partial w}{\partial y}, \quad u_z = w(x, y)\,. \tag{333}$$

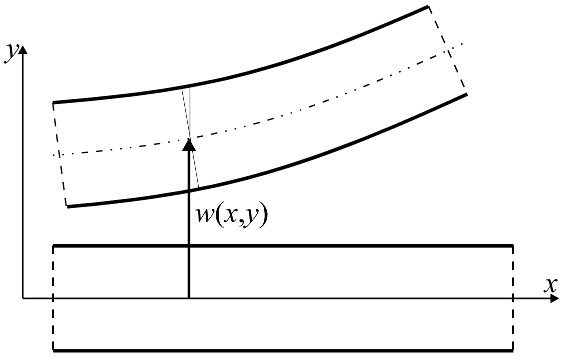


Figure 50. The bar segment deformation scheme.

Thus stains are expressed by the relations:

$$\varepsilon_x = \frac{\partial u_x}{\partial x} = -z\frac{\partial^2 w}{\partial x^2},\quad \varepsilon_y = \frac{\partial u_y}{\partial y} = -z\frac{\partial^2 w}{\partial y^2},\quad \gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = -2z\frac{\partial^2 w}{\partial x \partial y}. \tag{334}$$

The strain vector can be presented in the form:

$$\boldsymbol{\varepsilon} = -z\boldsymbol{\partial} w(x,y), \tag{335}$$

where vector $\boldsymbol{\partial}$ is the vector of differential operators:

$$\boldsymbol{\partial} = \begin{bmatrix} \partial_{xx} \\ \partial_{yy} \\ 2\partial_{xy} \end{bmatrix},\ \partial_{xx} = \frac{\partial^2}{\partial x^2},\ \partial_{yy} = \frac{\partial^2}{\partial y^2},\ \partial_{xy} = \frac{\partial^2}{\partial x \partial y}.$$

Let us assume that there is a plane stress condition in the plate, so the stress vector can be determined as follows:

$$\boldsymbol{\sigma} = \mathbf{D}\cdot\boldsymbol{\varepsilon} = -z\mathbf{D}\,\boldsymbol{\partial} w(x,y), \tag{336}$$

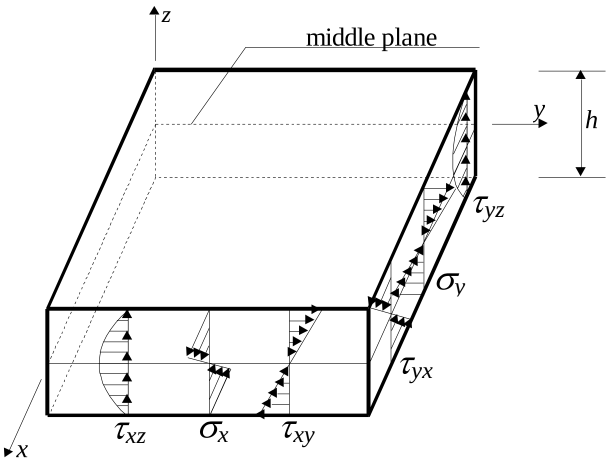
where **D** is the matrix of material constants determined for plane stress (Eqn. (13)).

Now we introduce in the expression of internal forces (moments and shearing forces – Figure 51)

$$M_x = \int_{-h/2}^{h/2} \sigma_x z dz,\quad M_y = \int_{-h/2}^{h/2} \sigma_y z dz,\quad M_{xy} = \int_{-h/2}^{h/2} \tau_{xy} z dz,$$

$$Q_x = \int_{-h/2}^{h/2} \tau_{xz} dz,\quad Q_y = \int_{-h/2}^{h/2} \tau_{yz} dz. \tag{337}$$

a) stresses



b) internal forces

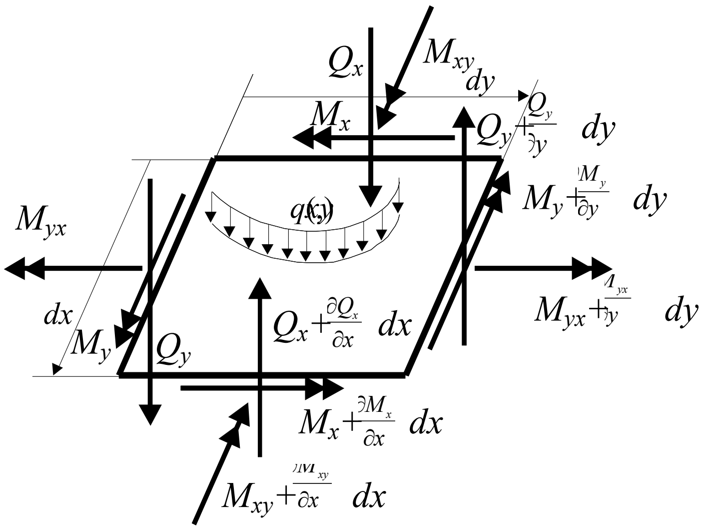


Figure 51. The distribution of stresses, external loads and internal forces in the plate element.

The equilibrium of an infinitesimal plate element shown in Figure 51b leads to the set of equations:

$$\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+q(x,y)=0\,,$$

$$\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}=Q_x\,, \tag{338}$$

$$\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_y}{\partial y}=Q_y\,.$$

After integration Eqn. (337) taking into consideration Eqn. (336), we obtain

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right),$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right), \tag{339}$$

$$M_{xy} = -D(1-\nu)\frac{\partial^2 w}{\partial x \partial y},$$

where D denotes the plate stiffness defined by the equation

$$D = \frac{Eh^3}{12\left(1-\nu^2\right)}. \tag{340}$$

From the last two Eqn. (338), we obtain relations for the shearing forces:

$$Q_x = -D\left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2}\right),$$

$$Q_y = -D\left(\frac{\partial^3 w}{\partial x^2 \partial y} + \frac{\partial^3 w}{\partial y^3}\right). \tag{341}$$

Inserting the above equation describing shearing forces into the first Eqn. (338) we obtain

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q(x,y)}{D}. \tag{342}$$

It is a biharmonic partial differential equation which should be satisfied by the function of deflection $w(x,y)$ within the plate. The following boundary conditions should be realised at the edges of the plate:

a) $w = 0$, $\frac{\partial w}{\partial n} = 0$ - on the fixed edge,

b) $w = 0$, $\frac{\partial^2 w}{\partial n^2} = 0$ - on the free supported edge,

c) $M_n = 0$, $V_n = 0$ - on the free edge.

In the above equations n defines the direction of the line which is perpendicular to the edge and V_n is the reduced force introduced by Kirchhoff in 1850, described by Timoshenko and Woinowsky-Krieger (1962). This force joins the influence of the torsion moment M_{ns} and the shearing force Q_n on the free edge Figure 51b:

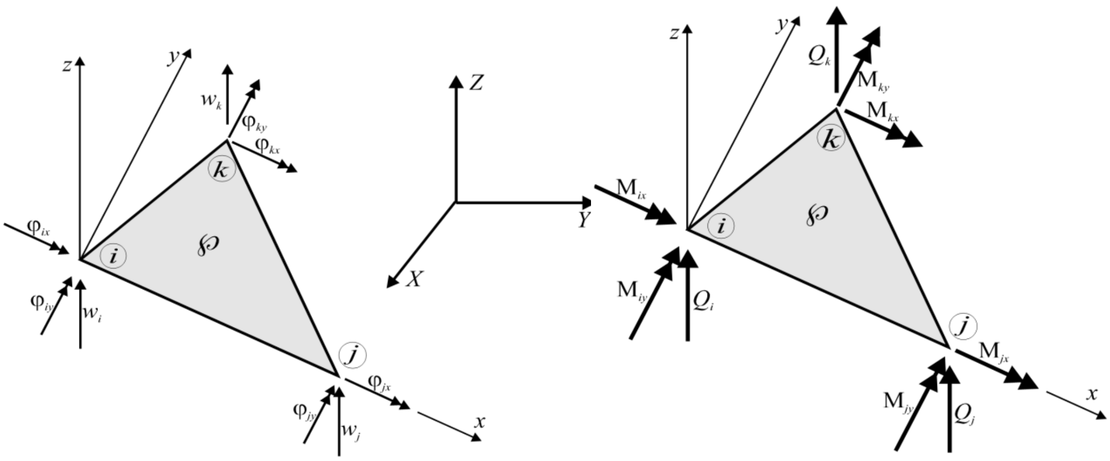
$$V_n = Q_n - \frac{\partial M_{ns}}{\partial s} = -D\left[\frac{\partial^3 w}{\partial n^3} + (2-\nu)\frac{\partial^3 w}{\partial n \partial s^2}\right] \tag{343}$$

where n describes the direction of the line which is perpendicular to the edge and s is the direction of the line which is parallel to the edge of the plate.

The modification of the boundary conditions is necessary here because the fourth order Eqn. (342) cannot be solved for three boundary conditions coming from the requirement of zero stress on the free edge: $M_{ns} = 0$, $M_n = 0$, $Q_n = 0$.

5.2. A finite triangular element of a thin plate

Now we show the way of building the stiffness matrix of a triangular element of a thin plate (Figure 52).



a) nodal displacements b) nodal forces

Figure 52. Nodal loads and displacements for the plate element in the local coordinate system.

We also introduce a few convenient notations:

- $w(x,y)$ stands for the function of displacement of the middle plane of an element;
- $\varphi_x = \dfrac{\partial w}{\partial y}$ is the rotation angle of the element about the x axis;
- $\varphi_y = -\dfrac{\partial w}{\partial x}$ is a rotation angle of the element about the y axis.

As seen in Figure 52, the node of a plate element has three degrees of freedom. Hence nodal displacement vectors of the element in the local system can be written as follows:

$$\mathbf{u}'_i = \begin{bmatrix} w_i \\ \varphi_{ix} \\ \varphi_{iy} \end{bmatrix}, \ \mathbf{u}'_j = \begin{bmatrix} w_j \\ \varphi_{jx} \\ \varphi_{jy} \end{bmatrix}, \ \mathbf{u}'_k = \begin{bmatrix} w_k \\ \varphi_{kx} \\ \varphi_{ky} \end{bmatrix} \tag{344}$$

and an element displacement vector:

$$\mathbf{u}'^e = \begin{bmatrix} \mathbf{u}'_i \\ \mathbf{u}'_j \\ \mathbf{u}'_k \end{bmatrix}. \tag{345}$$

Directions of both nodal displacements and forces (Figure 52b) are the same, so the nodal forces vectors have a similar notation:

$$\mathbf{f}'_i = \begin{bmatrix} Q_i \\ M_{ix} \\ M_{iy} \end{bmatrix}, \ \mathbf{f}'_j = \begin{bmatrix} Q_j \\ M_{jx} \\ M_{jy} \end{bmatrix}, \ \mathbf{f}'_k = \begin{bmatrix} Q_k \\ M_{kx} \\ M_{ky} \end{bmatrix}. \tag{346}$$

Hence we write the nodal force vector of the element as follows:

$$\mathbf{f}'^e = \begin{bmatrix} \mathbf{f}'_i \\ \mathbf{f}'_j \\ \mathbf{f}'_k \end{bmatrix}. \tag{347}$$

We approximate the surface of the deformed element by the polynomial of the third order proposed by J.L.Tocher in 1962:

$$w(x,y) = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2 + a_7 x^3 + a_8\left(x^2 y + xy^2\right) + a_9 y^3 = \boldsymbol{\eta}^{\mathsf{T}}\mathbf{a}, \tag{348}$$

where

$$\boldsymbol{\eta} = \begin{bmatrix} 1 \\ x \\ y \\ x^2 \\ xy \\ y^2 \\ x^3 \\ x^2 y + xy^2 \\ y^3 \end{bmatrix}, \qquad \mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \end{bmatrix}.$$

We determine the coefficients a_1 ... a_9 of the function $w(x,y)$ from the boundary conditions at the nodes i, j, k:

$$w(x_i, y_i) = w_i\,,\ \varphi_x(x_i, y_i) = \varphi_{ix}\,,\ \varphi_y(x_i, y_i) = \varphi_{iy}\,,$$

$$w(x_j, y_j) = w_j\,,\ \varphi_x(x_j, y_j) = \varphi_{jx}\,,\ \varphi_y(x_j, y_j) = \varphi_{jy}\,, \tag{349}$$

$$w(x_k, y_k) = w_k\,,\ \varphi_x(x_k, y_k) = \varphi_{kx}\,,\ \varphi_y(x_k, y_k) = \varphi_{ky}\,.$$

After calculating the rotation angles, we obtain

$$\begin{aligned} \varphi_x &= \frac{\partial w(x,y)}{\partial y} = a_3 + a_5 x + 2a_6 y + a_8(x^2 + 2xy) + 3a_9 y^2\,, \\ \varphi_y &= -\frac{\partial w(x,y)}{\partial x} = -\left[a_2 + 2a_4 x + a_5 y + 3a_7 x^2 + a_8(2xy + y^2)\right]. \end{aligned} \tag{350}$$

Now we insert Eqn. (348) and (350) into boundary conditions (349) obtaining:

$$\mathbf{M}\,\mathbf{a} = \mathbf{u}'^{e}\,, \tag{351}$$

where **M** is the square matrix dependent on nodal coordinates of the element.

$$
\mathbf{M}=\begin{array}{cc}
\begin{matrix} \quad a_1 & \;\; a_2 & \;\; a_3 & \;\; a_4 & \;\; a_5 & \;\; a_6 & \;\; a_7 & \qquad a_8 & \qquad a_9 \end{matrix} & \\
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & x_j & 0 & x_j^2 & 0 & 0 & x_j^3 & 0 & 0 \\
0 & 0 & 1 & 0 & x_j & 0 & 0 & x_j^2 & 0 \\
0 & -1 & 0 & -2x_j & 0 & 0 & -3x_j^2 & 0 & 0 \\
1 & x_k & y_k & x_k^2 & x_k y_k & y_k^2 & x_k^3 & x_k^2 y_k + x_k y_k^2 & y_k^3 \\
0 & 0 & 1 & 0 & x_k & 2y_k & 0 & x_k^2 + 2x_k y_k & 3y_k^2 \\
0 & -1 & 0 & -2x_k & -y_k & 0 & -3x_k^2 & -2x_k y_k - y_k^2 & 0
\end{bmatrix} &
\begin{matrix} w_i \\ \varphi_{ix} \\ \varphi_{iy} \\ w_j \\ \varphi_{jx} \\ \varphi_{jy} \\ w_k \\ \varphi_{kx} \\ \varphi_{ky} \end{matrix}
\end{array}
\tag{352}
$$

We can present the solution of Eqn. (351) as follows:

$$
\mathbf{a} = \mathbf{M}^{-1}\,\mathbf{u}'^{e}\,, \tag{353}
$$

where $\mathbf{M}^{-1}$ is the inverse matrix of $\mathbf{M}$. The solution of $\mathbf{M}^{-1}$ is possible when $\det \mathbf{M} \neq 0$ (comp. Appendix 1) which is not always the case in our problem because

$$
\det \mathbf{M} = x_j^5 y_k^5 \left(2x_k + y_k - x_j\right) \tag{354}
$$

It means that in cases when the node k of the element is on the line described by equation $y = x_j - 2x$, then the matrix $\mathbf{M}$ is singular. Thus, the problem is solved by changing the local coordinate system.

Now we calculate a strain vector determined by Eqn. (335).

$$
\boldsymbol{\varepsilon} = -z\boldsymbol{\partial} w(x,y) = -z\boldsymbol{\partial}\boldsymbol{\eta}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{u}'^{e} = -z\mathbf{B}^{*}\mathbf{M}^{-1}\mathbf{u}'^{e}, \tag{355}
$$

where $\mathbf{B}^{*} = \boldsymbol{\partial}\boldsymbol{\eta}^{\mathrm{T}}$ is a rectangular matrix of which components are equal to:

$$
\mathbf{B}^{*} = \begin{bmatrix}
0 & 0 & 0 & 2 & 0 & 0 & 6x & 2y & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 2x & 6y \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 4(x+y) & 0
\end{bmatrix}. \tag{356}
$$

Comparing Eqn. (355) with the definition of the geometric matrix $\mathbf{B}^{e}$ described by Eqn. (36) and (38), we obtain

$$\mathbf{B}^e = -z\,\mathbf{B}^*\mathbf{M}^{-1}\,. \tag{357}$$

Hence we can make use of the definition of the stiffness matrix contained in Eqn. (50):

$$\begin{aligned}\mathbf{K}'^e &= \int_{\mathcal{V}} \left(\mathbf{B}^e\right)^{\mathsf{T}} \mathbf{D}\mathbf{B}^e d\mathcal{V} = \left(\mathbf{M}^{-1}\right)^{\mathsf{T}} \int_{-h/2}^{h/2} z^2 dz \int_{\mathcal{A}} \left(\mathbf{B}^*\right)^{\mathsf{T}} \mathbf{D}\mathbf{B}^* d\mathcal{A}\,\mathbf{M}^{-1} = \\ &= \frac{Eh^3}{12\left(1-\nu^2\right)} \left(\mathbf{M}^{-1}\right)^{\mathsf{T}} \int_{\mathcal{A}} \left(\mathbf{B}^*\right)^{\mathsf{T}} \mathbf{D}\,\mathbf{B}^* d\mathcal{A}\,\mathbf{M}^{-1}\,.\end{aligned} \tag{358}$$

After denoting the integration in the above equation by $\mathbf{K}^*$ and applying the definition of plate stiffness we have

$$\mathbf{K}'^e = D\left(\mathbf{M}^{-1}\right)^{\mathsf{T}} \mathbf{K}^*\mathbf{M}^{-1}\,. \tag{359}$$

After calculating the matrix multiplication inside the integration in Eqn. (358), we have

$$\mathbf{K}^* = \int_{\mathcal{A}} \mathbf{S}\,d\mathcal{A}\,, \tag{360}$$

where

$$\mathbf{S} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 4\nu & 12x & 4\left(y+x\nu\right) & 12y\nu \\ 0 & 0 & 0 & 0 & 2\left(1-\nu\right) & 0 & 0 & 4\left(x+y\right)\left(1-\nu\right) & 0 \\ 0 & 0 & 0 & 4\nu & 0 & 4 & 12x\nu & 4\left(y\nu+x\right) & 12y \\ 0 & 0 & 0 & 12x & 0 & 12x\nu & 36x^2 & 12\left(xy+x^2\nu\right) & 36xy\nu \\ 0 & 0 & 0 & 4\left(y+x\nu\right) & 4\left(x+y\right)\left(1-\nu\right) & 4\left(y\nu+x\right) & 12\left(xy+x^2\nu\right) & \begin{matrix} 4\left(3y^2+4xy+3x^2\right)+ \\ -8\nu\left(x^2+xy+y^2\right) \end{matrix} & 12\left(y^2\nu+xy\right) \\ 0 & 0 & 0 & 12y\nu & 0 & 12y & 36xy\nu & 12\left(y^2\nu+xy\right) & 36y^2 \end{bmatrix}.$$

While calculating the integration of functions exiting in Eqn. (360), the following relations are helpful:

$$\begin{aligned}&\int_{\mathcal{A}} d\mathcal{A} = \frac{1}{2} x_j y_k\,, \\ &\int_{\mathcal{A}} x\,d\mathcal{A} = \frac{1}{6} x_j y_k \left(x_j + x_k\right),\end{aligned} \tag{361}$$

$$\int_{\mathscr{A}} y d\mathscr{A} = \frac{1}{6} x_j y_k^2 ,$$

$$\int_{\mathscr{A}} x^2 d\mathscr{A} = \frac{1}{12} x_j y_k \left(x_j^2 + x_j x_k + x_k^2 \right),$$

$$\int_{\mathscr{A}} xy d\mathscr{A} = \frac{1}{24} x_j y_k^2 \left(x_j + 2x_k \right),$$

$$\int_{\mathscr{A}} y^2 d\mathscr{A} = \frac{1}{12} x_j y_k^2 .$$

Matrix Eqn. (358) is determined in the local coordinate system. We have to transform it to the global coordinate system in accordance with relation Eqn. (53):

$$\mathbf{K}^e = \mathbf{R}^e \mathbf{K}'^e \left(\mathbf{R}^e \right)^{\mathrm{T}} .$$

The rotation matrix of an element $\mathbf{R}^e$ is equal to:

$$\mathbf{R}^e = \begin{bmatrix} \mathbf{R}_i & & \\ & \mathbf{R}_j & \\ & & \mathbf{R}_k \end{bmatrix}, \tag{362}$$

where $\mathbf{R}_i$, $\mathbf{R}_j$, $\mathbf{R}_k$ are the transformation matrices of nodes. If we use the same coordinate systems for all nodes (it has been done in this chapter), then we can use only one transformation matrix: $\mathbf{R}_j = \mathbf{R}_i$, $\mathbf{R}_k = \mathbf{R}_i$,

$$\mathbf{R}_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix}, \tag{363}$$

where $c = \cos\alpha$, $s = \sin\alpha$ and α is the angle between the *X* axis of the global system and the *x* axis of the local system (Figure 53). Value 1 in the first row of the matrix $\mathbf{R}_i$ is the consequence of a fact that axes *Z* and *z* are parallel.

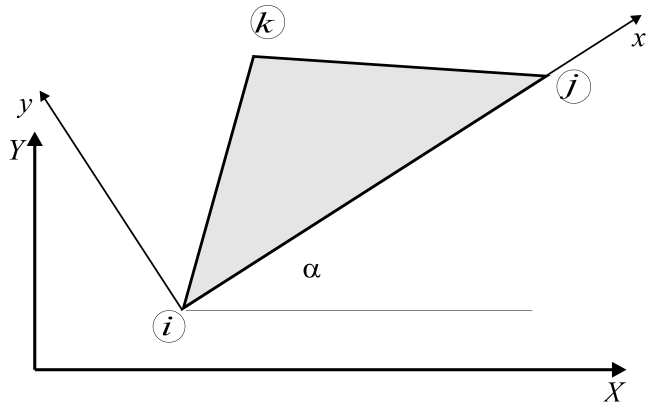


Figure 53. The plate arrangement in the global coordinate system.

The triangular element for which the matrix stiffness has been obtained has a convenient feature. Namely, it allows us to discrete plates of any shape without any difficulty. This element joined with a 2D triangular element can be used as a shell element (comp. Rakowski and Kacprzyk (1993)).

Elements of any other shapes (rectangular or quadrilateral) are presented in the books written by Bathe (1996), Rakowski and Kacprzyk (1993), Rao (1982) or Zienkiewicz (1972, 1994).

5.3. A triangular element of a thin shell

As it has been noted at the previous point, an element containing 2D triangular and plate elements can be used as a shell element. Approximating a curved surface (which is the middle surface of a shell) with the help of plate elements reminds the simplification we apply to approach the arc with the help of a broken line. We intuitively feel that the smaller the curve line segments are, the better they replace the curve axis of the arc (Figure 54). Similarly the smaller the plane shell element dimensions and the smaller β angles (comp. Figure 54) of neighbouring elements are, the better this element describes displacements and internal forces in the structure. Detailed calculations and experiments confirm the correctness of this approximation (comp. Zienkiewicz (1972)).

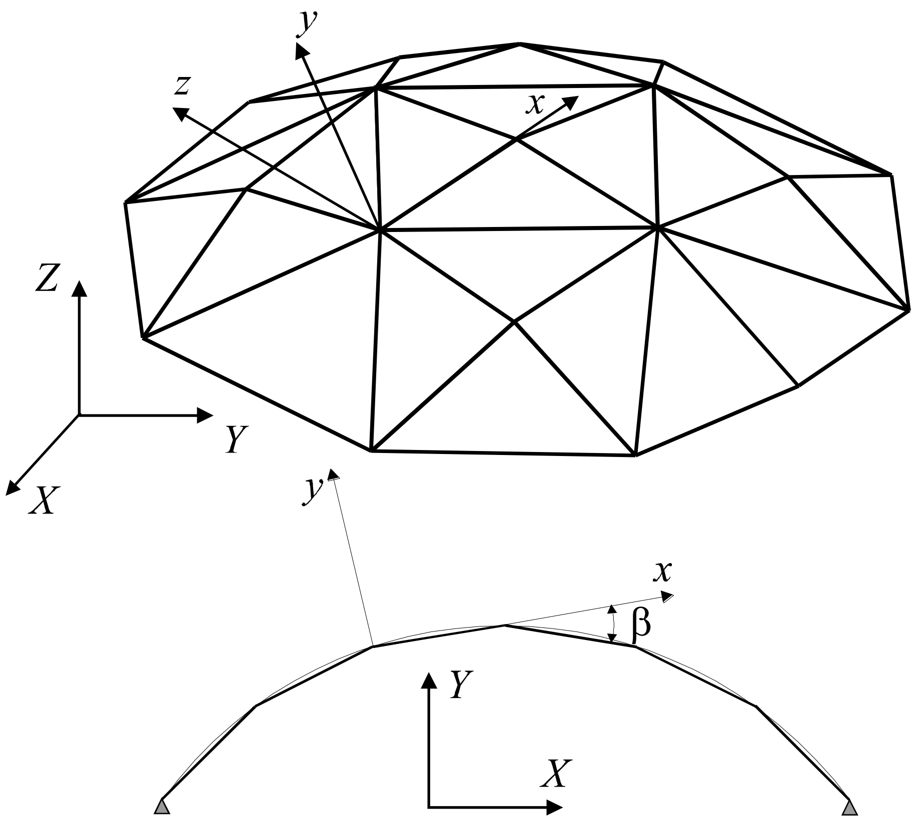


Figure 54. The exemplary shell division into finite elements.

Connecting displacement and internal force vectors of the triangular elements described by Eqn. (292), (344), (296) and (346), we obtain shell element nodes possessing five degrees of freedom:

$$\mathbf{u}'_i = \begin{bmatrix} u_{ix} \\ u_{iy} \\ u_{iz} \\ \varphi_{ix} \\ \varphi_{iy} \end{bmatrix}, \qquad \mathbf{f}'_i = \begin{bmatrix} F_{ix} \\ F_{iy} \\ F_{iz} \\ M_{ix} \\ M_{iy} \end{bmatrix}. \tag{364}$$

In Eqn. (364), u_{iz} and F_{iz} denote, respectively, a nodal displacement and a force parallel to the z axis of the local coordinate system. In Eqn. (344) and (346), these values are marked with w_i and Q_i (comp. Figure 52 and Figure 55).

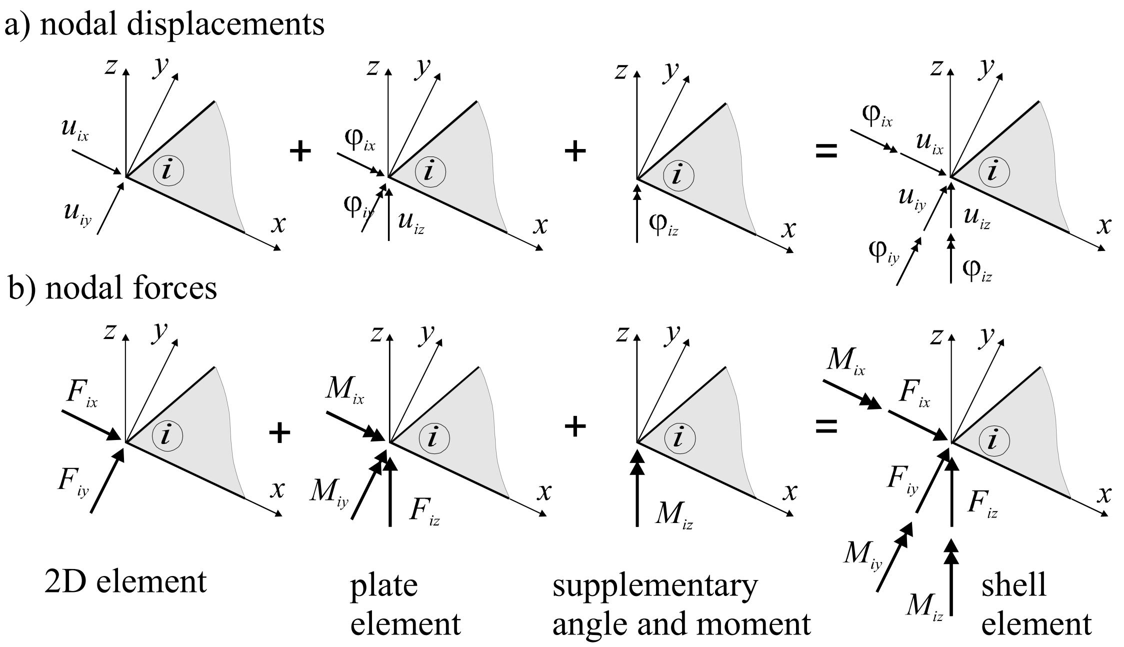


Figure 55. The shell element composition of a 2D and plate elements.

Simplifying the description of a node movement by disregarding the rotation around the axis perpendicular to the element leads to the singularity of the shell stiffness matrix modelled by the elements mentioned before. This difficulty is solved by assuming three components of the rotation and moment vectors which requires the evaluation of the plate element torsional stiffness.

Since the torsional stiffness is not important in shell statics and dynamics problems, the fictitious value of this stiffness is often assumed (comp. Zienkiewicz (1972)). Hence the dependence between the torsional moments and angles can be presented as a variable independent of other nodal forces and displacements of an element:

$$\begin{bmatrix} M_{iz} \\ M_{jz} \\ M_{kz} \end{bmatrix} = \alpha EhA \begin{bmatrix} 1 & -0.5 & -0.5 \\ -0.5 & 1 & -0.5 \\ -0.5 & -0.5 & 1 \end{bmatrix} \begin{bmatrix} \varphi_{iz} \\ \varphi_{jz} \\ \varphi_{kz} \end{bmatrix}. \tag{365}$$

In the above relationship suggested by Zienkiewicz (1972), E is Young's modulus, h is the element thickness, A is the area of a cross section and α denotes an indemensional coefficient which is so small that it does not have any significant influence on the solution of a set of equations. The value of this coefficient is assumed within the range 0.01÷0.001, Zienkiewicz (1972, 1994) suggests taking the value equal to 0.03 or less.

Surveying the described matrices, we obtain the stiffness matrix of the triangular shell element nodes having six degrees of freedom:

$$\mathbf{f}'^{e} = \mathbf{K}'^{e}\,\mathbf{u}'^{e} \tag{366}$$

or

$$\begin{bmatrix} \mathbf{f}'_i \\ \mathbf{f}'_j \\ \mathbf{f}'_k \end{bmatrix} = \begin{bmatrix} \mathbf{K}'_{ii} & \mathbf{K}'_{ij} & \mathbf{K}'_{ik} \\ \mathbf{K}'_{ji} & \mathbf{K}'_{jj} & \mathbf{K}'_{jk} \\ \mathbf{K}'_{ki} & \mathbf{K}'_{kj} & \mathbf{K}'_{kk} \end{bmatrix} \begin{bmatrix} \mathbf{u}'_i \\ \mathbf{u}'_j \\ \mathbf{u}'_k \end{bmatrix} \tag{367}$$

where $\mathbf{f}'_i$ and $\mathbf{u}'_i$ denote full vectors of nodal forces and displacements:

$$\mathbf{u}'_i = \begin{bmatrix} u_{ix} \\ u_{iy} \\ u_{iz} \\ \varphi_{ix} \\ \varphi_{iy} \\ \varphi_{iz} \end{bmatrix}, \qquad \mathbf{f}'_i = \begin{bmatrix} F_{ix} \\ F_{iy} \\ F_{iz} \\ M_{ix} \\ M_{iy} \\ M_{iz} \end{bmatrix}. \tag{368}$$

Every block of the stiffness matrix in Eqn. (367) consists of '2D element', 'plate' and 'torsional' parts (Eqn. (365))

$$\begin{bmatrix} F_{ix} \\ F_{iy} \\ F_{iz} \\ M_{ix} \\ M_{iy} \\ M_{iz} \end{bmatrix} = \left[\begin{array}{cc|ccc|c} \multicolumn{2}{c|}{\multirow{2}{*}{${}^{\mathrm{t}}\mathbf{K}_{ij}$}} & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & & & & \\ 0 & 0 & & {}^{\mathrm{p}}\mathbf{K}_{ij} & & \\ 0 & 0 & & & & \\ \hline 0 & 0 & 0 & -0.5\,a_s & -0.5\,a_s & a_s \end{array}\right] \begin{bmatrix} u_{jx} \\ u_{jy} \\ u_{jz} \\ \varphi_{jx} \\ \varphi_{jy} \\ \varphi_{jz} \end{bmatrix} \tag{369}$$

where $a_s=\alpha EhA$ describes the fictitious torsional stiffness existing in Eqn. (365), ${}^{\mathrm{t}}\mathbf{K}_{ij}$ is the stiffness matrix block for the plate element (Eqn. (359)).

Transformation of this matrix to the global coordinate system can be done in the way described in Chapter 5 (p.5.2.2, p.5.2.3) in which we present the transformation of the stifness matrix of a 3D frame element with nodes having six degrees of freedom just as the nodes of a shell element. The method of obtaining the components of the rotate matrix described at point 5.2.3 is suitable for the triangular shell element whose *i* and *j*nodes determine the direction of the local *x* axis and the third *k* node can be a directional point.

The shell element described above is the simplest element which enables us to solve any shell statics problem. There certainly are more complex elements, both plane and space elements with at least four nodes described in books devoted to this subject as Irons and Ahmad (1980), Rakowski and Kacprzyk (1993), Rao (1982), Zienkiewicz (1972, 1994). We must remember about the possibility of significant simplification of a shell element description in case of axisymmetric structures. It is also possible to use cone or curvelinear elements with nodes having three degrees of freedom (Rakowski and Kacprzyk (1993), Zienkiewicz (1972, 1994)).

6. Brick elements

Brick is the three-dimensional element, which can be defined as a body, to which all dimensions are of the same order. The shape of the body and the load is any of available. With brick elements, fully 3D solid constructions can be modeled. Brick elements can replace every other type of element, like frame, shell and plate elements. A solid model is divided into brick elements and the geometric shape of the elements can be tetrahedron, hexahedron or prism with triangle base, which means, that a brick element is build of triangles and quadrilateral. Typical 3D shapes of elements are shown in Figure 56.

a) b) c)

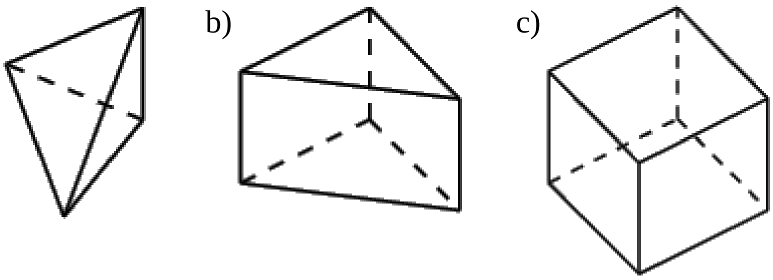


Figure 56. Three-dimensional shapes of brick elements. a) four-nodes (tetrahedron), b) six-nodes, c) eight-nodes (hexahedron).

In this chapter we will show how the tetrahedron is regarded in Finite Element Method. In Figure 57 you can see how displacements and nodal forces are located.

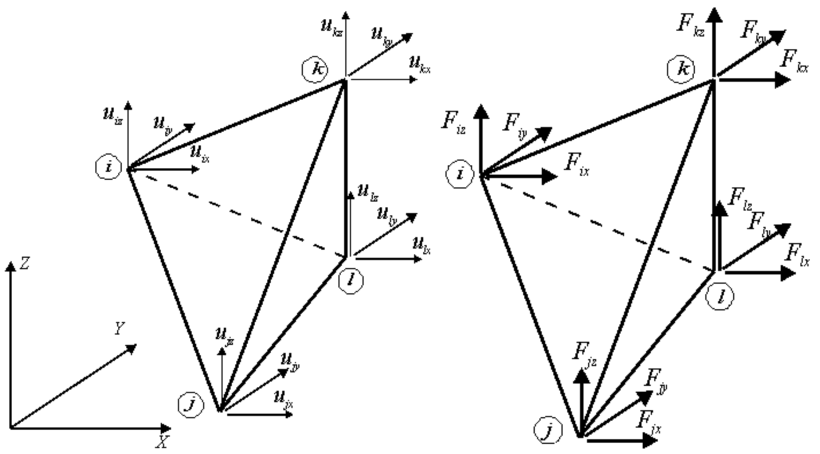


Figure 57. Nodal forces and displacements for the 3D element in the global coordinate system.

As it can be seen, in each node there are three displacements and forces in all global directions, but there are no rotations and moments. Vector of movements of element nodes and nodal forces can be written as follows:

$$\mathbf{u}'^{\mathbf{e}} = \begin{bmatrix} u_{ix} \\ u_{iy} \\ u_{iz} \\ u_{jx} \\ u_{jy} \\ u_{jz} \\ u_{kx} \\ u_{ky} \\ u_{kz} \\ u_{lx} \\ u_{ly} \\ u_{lz} \end{bmatrix}, \quad \mathbf{f}'^{\mathbf{e}} = \begin{bmatrix} F_{ix} \\ F_{iy} \\ F_{iz} \\ F_{jx} \\ F_{jy} \\ F_{jz} \\ F_{kx} \\ F_{ky} \\ F_{kz} \\ F_{lx} \\ F_{ly} \\ F_{lz} \end{bmatrix} \tag{370}$$

6.1. Relation between strain, stress and displacements

The brick works in a spatial state of stress.

In Figure 46 components of stress tensor are shown. There are three components of normal stresses σ_{xx}, σ_{yy}, σ_{zz}, and six sheer stresses, but according to the symmetry of a stress tensor components of shear stress we have $\tau_{xy}= \tau_{yx}$, $\tau_{xz}= \tau_{zx}$, and $\tau_{zy}= \tau_{yz}$, thus we have six independent components of stress which are composed in the stress vector:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix}, \tag{371}$$

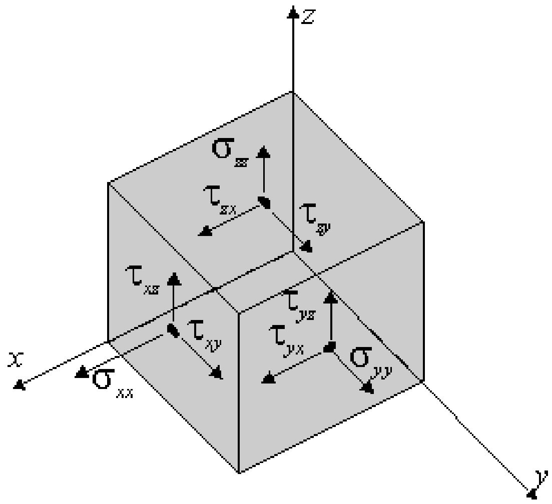


Figure 58. Stress tensor components.

Because of the fact that the brick works in three-dimensional state of stress and strain, the strain vector is similar to stress vector. Relationships between the components of displacement and strain vectors are similar to those for 2D elements:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u_x}{\partial x} \\ \dfrac{\partial u_y}{\partial y} \\ \dfrac{\partial u_z}{\partial z} \\ \dfrac{\partial u_x}{\partial y} + \dfrac{\partial u_y}{\partial x} \\ \dfrac{\partial u_y}{\partial z} + \dfrac{\partial u_z}{\partial y} \\ \dfrac{\partial u_z}{\partial x} + \dfrac{\partial u_x}{\partial z} \end{bmatrix}, \tag{372}$$

The relationship between vectors **σ** and **ε**is, like in two-dimensional elements, described by constitutive equations.For elastic isotropic materials the constitutive equation is shown below:

$$\boldsymbol{\sigma} = \mathbf{D}\cdot\boldsymbol{\varepsilon}, \tag{373}$$

where $\mathbf{D}$ is the square matrix with dimensions 6×6 containing the material constants, described in Chapter 1, shown in Eqn. (7).

6.2. Stiffness matrix of 3D element

The stiffness matrix is the same as for all previous chapters:

$$\mathbf{f}^{e} = \mathbf{K}^{e}\,\mathbf{u}^{e}$$

where $\mathbf{K}^{e}$ has the same formula as in Chapter 6:

$$\mathbf{K}^{e} = \int_{\mathcal{V}} \left(\mathbf{B}^{e}\right)^{\mathrm{T}} \mathbf{D}\mathbf{B}^{e} d\mathcal{V}\ . \tag{374}$$

In case of brick element $\mathbf{B}^{e}$ has following formula:

$$\mathbf{B}^{e} = \mathcal{D}\mathbf{N} \tag{375}$$

where $\mathbf{N}$ is matrix of shape function (which will be described later), $\mathcal{D}$ is a matrix of differential operators:

$$\mathcal{D} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix}. \tag{376}$$

6.3. Shape function of 3D elements

Displacement of every point of brick element in three-dimensional coordinate system can be written as a function:

$$\begin{aligned} u_x(x,y,z) &= a_1 + a_2 x + a_3 y + a_4 z\ , \\ u_y(x,y,z) &= b_1 + b_2 x + b_3 y + b_4 z\ , \\ u_z(x,y,z) &= c_1 + c_2 x + c_3 y + c_4 z\ . \end{aligned} \tag{377}$$

These are displacements in every of three dimensions located in any point with coordinates (x, y, z). All functions are linear.

Eqn. (377) can be written as a function depending on displacements in every of four nodes in tetrahedral element. Designations are related to Figure 57:

$$u_x(x,y,z) = N_i(x,y,z)u_{ix} + N_j(x,y,z)u_{jx} + N_k(x,y,z)u_{kx} + N_l(x,y,z)u_{lx}\,,$$
$$u_y(x,y,z) = N_i(x,y,z)u_{iy} + N_j(x,y,z)u_{jy} + N_k(x,y,z)u_{ky} + N_l(x,y,z)u_{ly}\,, \tag{378}$$
$$u_z(x,y,z) = N_i(x,y,z)u_{iz} + N_j(x,y,z)u_{jz} + N_k(x,y,z)u_{kz} + N_l(x,y,z)u_{lz}\,.$$

where:

$$N_i(x,y,z) = a_i + b_i x + c_i y + d_i z\,,$$
$$N_j(x,y,z) = a_j + b_j x + c_j y + d_j z\,,$$
$$N_k(x,y,z) = a_k + b_k x + c_k y + d_k z\,, \tag{379}$$
$$N_l(x,y,z) = a_l + b_l x + c_l y + d_l z\,.$$

Eqn. (378) can be written in matrix form:

$$\mathbf{u}(x,y,z) = \mathbf{N}\mathbf{u}'^{e} \tag{380}$$

where $\mathbf{u}(x, y, z)$ is a displacement vector of any point located inside the brick element, $\mathbf{u}'^{e}$ is nodal displacement vector (Eqn. (370)), and **N** is stiffness function matrix.

$$\mathbf{u}(x,y,z) = \begin{bmatrix} u_x(x,y,z) \\ u_y(x,y,z) \\ u_z(x,y,z) \end{bmatrix}, \tag{381}$$

$$\mathbf{N} = \begin{bmatrix} N_i & 0 & 0 & N_j & 0 & 0 & N_k & 0 & 0 & N_l & 0 & 0 \\ 0 & N_i & 0 & 0 & N_j & 0 & 0 & N_k & 0 & 0 & N_l & 0 \\ 0 & 0 & N_i & 0 & 0 & N_j & 0 & 0 & N_k & 0 & 0 & N_l \end{bmatrix}. \tag{382}$$

Shape function for element of eight nodes can be written for any node as bellow:

$$N_i = \frac{1}{8}(1+\xi_o)(1+\eta_o)(1+\zeta_o)\,, \tag{383}$$

where

$$\xi = \frac{x}{l_x}, \quad \eta = \frac{y}{l_y}, \quad \zeta = \frac{z}{l_z}. \tag{384}$$

This shape function is based on Lagrangian interpolation for the three variables of function passing through two points. Designations for eight-node element are shown in Figure 59.

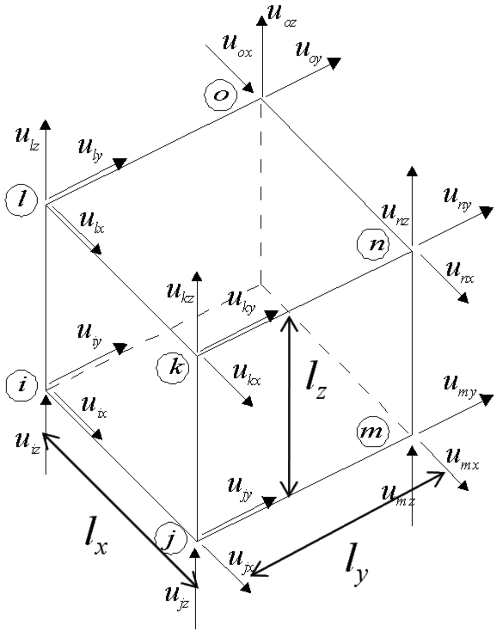


Figure 59. Displacements for cubic brick element.

Now, when we have the shape function, we can return Eqn. (375), where the geometric matrix for nodes $\mathbf{B}^e$ can be expressed for four-node element:

$$\mathbf{B}^e = \mathscr{D}\mathbf{N} = \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 & 0 & \frac{\partial N_j}{\partial x} & 0 & 0 & \frac{\partial N_k}{\partial x} & 0 & 0 & \frac{\partial N_l}{\partial x} & 0 & 0 \\ 0 & \frac{\partial N_i}{\partial y} & 0 & 0 & \frac{\partial N_j}{\partial y} & 0 & 0 & \frac{\partial N_k}{\partial y} & 0 & 0 & \frac{\partial N_l}{\partial y} & 0 \\ 0 & 0 & \frac{\partial N_i}{\partial z} & 0 & 0 & \frac{\partial N_j}{\partial z} & 0 & 0 & \frac{\partial N_k}{\partial z} & 0 & 0 & \frac{\partial N_l}{\partial z} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} & 0 & \frac{\partial N_j}{\partial y} & \frac{\partial N_j}{\partial x} & 0 & \frac{\partial N_k}{\partial y} & \frac{\partial N_k}{\partial x} & 0 & \frac{\partial N_l}{\partial y} & \frac{\partial N_l}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial z} & \frac{\partial N_i}{\partial y} & 0 & \frac{\partial N_j}{\partial z} & \frac{\partial N_j}{\partial y} & 0 & \frac{\partial N_k}{\partial z} & \frac{\partial N_k}{\partial y} & 0 & \frac{\partial N_l}{\partial z} & \frac{\partial N_l}{\partial y} \\ \frac{\partial N_i}{\partial z} & 0 & \frac{\partial N_i}{\partial y} & \frac{\partial N_j}{\partial z} & 0 & \frac{\partial N_j}{\partial x} & \frac{\partial N_k}{\partial z} & 0 & \frac{\partial N_k}{\partial x} & \frac{\partial N_l}{\partial z} & 0 & \frac{\partial N_l}{\partial x} \end{bmatrix} \tag{385}$$

By substituting Eqn. (379) into Eqn. (385) we can obtain a simplified form:

$$\mathbf{B}^e = \begin{bmatrix} b_i & 0 & 0 & b_j & 0 & 0 & b_k & 0 & 0 & b_l & 0 & 0 \\ 0 & c_i & 0 & 0 & c_j & 0 & 0 & c_k & 0 & 0 & c_l & 0 \\ 0 & 0 & d_i & 0 & 0 & d_j & 0 & 0 & d_k & 0 & 0 & d_l \\ c_i & b_i & 0 & c_j & b_j & 0 & c_k & b_k & 0 & c_l & b_l & 0 \\ 0 & d_i & c_i & 0 & d_j & c_j & 0 & d_k & c_k & 0 & d_l & c_l \\ d_i & 0 & b_i & d_j & 0 & b_j & d_k & 0 & b_k & d_l & 0 & b_l \end{bmatrix}. \tag{386}$$

6.4. Strain and stress in element on tetrahedron example

Now after knowing the geometric matrix for tetrahedral element, the strain vector can be obtained from the following equation:

$$\boldsymbol{\varepsilon} = \mathscr{D}\mathbf{N}(x, y, z)\mathbf{u}'^{e} = \mathbf{B}^{e}\mathbf{u}'^{e}, \tag{387}$$

which is

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \begin{bmatrix} b_i & 0 & 0 & b_j & 0 & 0 & b_k & 0 & 0 & b_l & 0 & 0 \\ 0 & c_i & 0 & 0 & c_j & 0 & 0 & c_k & 0 & 0 & c_l & 0 \\ 0 & 0 & d_i & 0 & 0 & d_j & 0 & 0 & d_k & 0 & 0 & d_l \\ c_i & b_i & 0 & c_j & b_j & 0 & c_k & b_k & 0 & c_l & b_l & 0 \\ 0 & d_i & c_i & 0 & d_j & c_j & 0 & d_k & c_k & 0 & d_l & c_l \\ d_i & 0 & b_i & d_j & 0 & b_j & d_k & 0 & b_k & d_l & 0 & b_l \end{bmatrix} \cdot \mathbf{u}'^{e} \tag{388}$$

where the displacement vector $\mathbf{u}'^{e}$ is described by Eqn. (370)

Knowing the strain vector it can be used in Eqn. (288) to obtain the stress vector:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-\nu} & 1 & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} \tag{389}$$

6.5. Rules of FEM mesh formation for 3D brick models

Exemplary of the 3D models are shown in Figure 60.

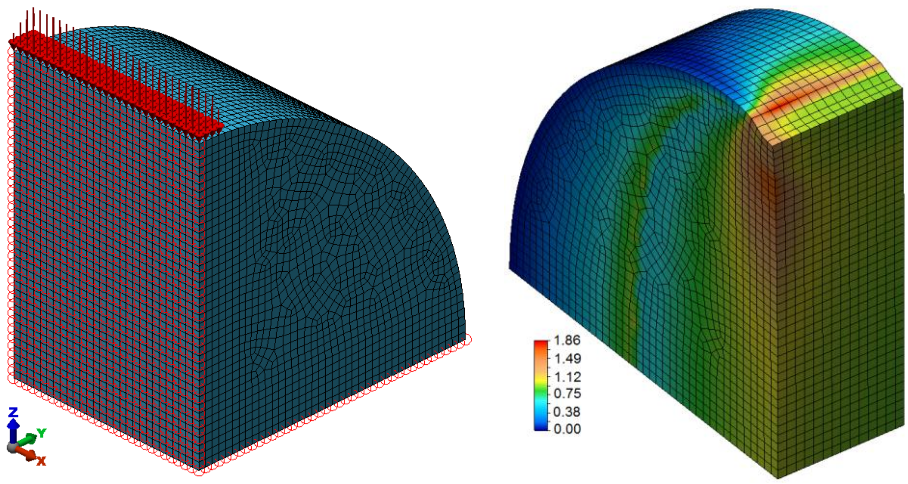


Figure 60. Example of the3D brick elements use.

There are some rules that are need to be followed:

- FEM grid must take into account the shape of the structure,
- in a hole in the model, nodes must be located so that there is no possibility to create a FEM element, but to form an empty area,
- mesh nodes must be located in location of concentrated loads,
- mesh nodes must be located in the boundary condition points,
- the edges of the grid must be located on the border between parts of elements made of different materials,
- if the job is symmetrical, the mesh also should be symmetrical.

FEM mesh should be concentrated in areas of high stress concentration and in areas of rapid change in stress (high value of gradient). Such areas are located:

- at the corners,
- around the points of application of concentrated forces,
- around the supports,

If the component is narrow then in cross section there should be at least four belts of elements. Only then they will be able to describe the change in stress in the cross section.

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Appendix 1. Matrix Algebra

In this appendix we give the most important definitions of matrix algebra and we elaborate some functions and transformations of matrices which are most helpful in numerical applications and particularly in the finite element method.

Definitions

- Scalar - value determined only by its magnitude which can be expressed by a real number. The typical scalar values are mass, temperature, time, length, etc. We will denote the scalars by letters written in italic font.
- Vector - value determined by its modulus, direction and sense. The examples of vectors are force, displacement, velocity and rotation. We will denote the vectors by small letters written in bold font.
- Matrix- table containing most often scalars but it can also contain vectors or other matrices. Elements of a matrix are called components. It is a very convenient form of presentation of large quantities of data which we deal with in numerical methods. One of a matrix notation which we apply in this book looks as follows:

$$\mathbf{A} = \left[A_{ij}\right] = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{bmatrix}.$$

  We will denote quadratic matrices (they have the same number of columns and rows) and rectangular matrices (they have a different number of columns and rows) by capital letters written in bold font.
- Column matrix - will also be called a vector and it contains only one column. We will denote it just as vectors.
- Identity matrix - square matrix components of which are equal to zero except for those lying on the main diagonal (diagonal elements). Diagonal elements are equal to 1. We will mark the identity matrix by the capital letter **I** and in some cases by an index pointing dimensions of a matrix:

$$\mathbf{I}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

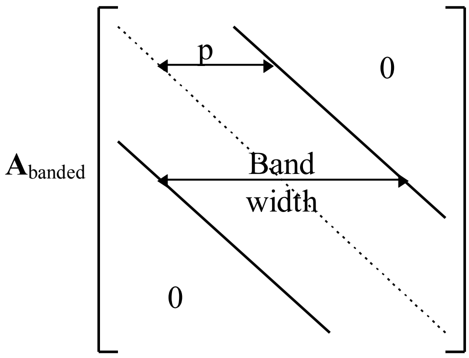
The components of the identity matrix can be written with the help of Kronecker's delta $\mathbf{I} = \left[\delta_{ij}\right]$ where $\delta_{ii} = 1$, $\delta_{ij} = 0$, when $i \neq j$.

- Triangular matrix - matrix containing either components equal to zero (**L**-triangular lower matrix) lying over the main diagonal or components also equal to zero (**U**-triangular upper matrix) lying below the main diagonal

$$\mathbf{L} = \left[L_{ij}\right] = \begin{bmatrix} L_{11} & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 \\ L_{31} & L_{32} & L_{33} & 0 \\ L_{41} & L_{42} & L_{43} & L_{44} \end{bmatrix},$$

$$\mathbf{U} = \left[U_{ij}\right] = \begin{bmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ 0 & U_{22} & U_{23} & U_{24} \\ 0 & 0 & U_{33} & U_{34} \\ 0 & 0 & 0 & U_{44} \end{bmatrix}.$$

- Band matrix - matrix containing components which are different from zero only when they are close to the main diagonal



p - width of half of the band

After suitable grouping of equilibrium equations, stiffness matrices are band matrices in the finite element method.

- Symmetric matrix - matrix with components satisfying the equation:

$$\mathbf{A}_{\text{sym}} \rightarrow \left[A_{ij}\right] = \left[A_{ji}\right]$$

Stiffness matrices are symmetric matrices in the finite element method.

- Transpose matrix - matrix in which we group components so that columns become rows:

$$\mathbf{B} = \mathbf{A}^{\mathsf{T}} \rightarrow \left[B_{ij}\right] = \left[A_{ji}\right].$$

Transpose matrices are denoted by the normal capital letter T which is written as an upper index.

- The main diagonal of a matrix is the diagonal which passes from the component A_{11} along other components having equal indices of a column and a row; that is A_{22} ... A_{ii} ... A_{nn}.

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}$$

Main diagonal

Matrix addition and subtraction

The operation of matrix addition is defined as follows:

$$\mathbf{C} = \mathbf{A} + \mathbf{B} \rightarrow C_{ij} = A_{ij} + B_{ij} \, ,$$

which means that the components of the matrix **C** resulting from the addition of matrices **A** and **B** are sums of suitable terms of matrices **A** and **B**. The matrix addition is possible only if both matrices (**A** and **B**) have the same number of columns and rows. The addition is a commutative operation:

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \, .$$

Similarly, we define matrix subtraction:

$$\mathbf{D} = \mathbf{A} - \mathbf{B} \rightarrow D_{ij} = A_{ij} - B_{ij} \, .$$

Example No 1.

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 8 & 2 \\ 2 & 4 & 1 & -2 \\ -1 & 0 & 3 & 4 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 3 & 2 & 5 & 1 \\ 0 & 2 & 1 & 3 \end{bmatrix},$$

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = \begin{bmatrix} (1+0) & (3+2) & (8+1) & (2+0) \\ (2+3) & (4+2) & (1+5) & (-2+1) \\ (-1+0) & (0+2) & (3+1) & (4+3) \end{bmatrix} = \begin{bmatrix} 1 & 5 & 9 & 2 \\ 5 & 6 & 6 & -1 \\ -1 & 2 & 4 & 7 \end{bmatrix},$$

$$\mathbf{D} = \mathbf{A} - \mathbf{B} = \begin{bmatrix} (1-0) & (3-2) & (8-1) & (2-0) \\ (2-3) & (4-2) & (1-5) & (-2-1) \\ (-1-0) & (0-2) & (3-1) & (4-3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 7 & 2 \\ -1 & 2 & -4 & -3 \\ -1 & -2 & 2 & 1 \end{bmatrix}.$$

Multiplication of a matrix by a scalar (scaling of a matrix)

Scaling a matrix is the name of an operation carried on its components and defined as follows:

$$\mathbf{E} = \alpha\mathbf{A} \rightarrow E_{ij} = \alpha A_{ij} \, ,$$

which means that components of the matrix **E** resulting from the multiplication of the matrix **A** by the scalar α are products of components of the matrix **A** and the value α.

Example No 2.

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 8 & 2 \\ 2 & 4 & 1 & -2 \\ -1 & 0 & 3 & 4 \end{bmatrix}, \qquad \alpha=3.5,$$

$$\mathbf{E} = 3.5\mathbf{A} = \begin{bmatrix} 3.5 & 10.5 & 28.0 & 7.0 \\ 7.0 & 14.0 & 3.5 & -7.0 \\ -3.5 & 0.0 & 10.5 & 14.0 \end{bmatrix}.$$

The matrix **E** which is the result of scaling has the same number of columns and rows just as the matrix **A** does.

Matrix multiplication

Let **C** be the result of multiplication of matrices **A** and **B**:

$$\mathbf{C} = \mathbf{A} \times \mathbf{B},$$

then components of the matrix **C** are results of the multiplication of rows of the matrix **A** by columns of the matrix **B** which can be written as follows:

$$C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj} ,$$

where n is the number of columns of the matrix **A**. As it is seen the multiplication of the matrices **A** and **B** is possible to perform if the number of columns of the matrix **A** is equal to the number of rows of the matrix **B**. The matrix **C** which is the result of multiplication has the number of rows equal to the number of rows of the matrix **A** and the number of columns equal to the number of columns of the matrix **B**.

$$
\begin{array}{cc}
 & \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1j} & B_{1m} \\ B_{21} & B_{22} & \dots & B_{2j} & B_{2m} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ B_{n1} & B_{n2} & \dots & B_{nj} & B_{nm} \end{bmatrix} \\
\mathbf{C} = \mathbf{A} \; \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ A_{i1} & A_{i2} & \dots & A_{in} \end{bmatrix} & C_{ij}
\end{array}
$$

Example No 3.

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 8 & 2 \\ 2 & 4 & 1 & -2 \\ -1 & 0 & 3 & 4 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 3 & 2 & 5 & 1 \\ 0 & 2 & 1 & 3 \end{bmatrix},$$

$$\mathbf{C} = \mathbf{AB}^{\top}$$

$\mathbf{AB}^{\top} =$

				0	3	0
				2	2	2
				1	5	1
				0	1	3
1	3	8	2	1·0+3·2+8·1+2·0= =14	1·3+3·2+8·5+2·1= =51	1·0+3·2+8·1+2·3= =20
2	4	1	-2	2·0+4·2+1·1-2 ·0= =9	2·3+4·2+1·5-2·1= =17	2·0+4·2+1·1-2·3= =3
-1	0	3	4	1·0+0·2+3·1+4·0= =3	1·3+0·2+3·5+4·1= =16	1·0+0·2+3·1+4·3= =15

$$\mathbf{C} = \begin{bmatrix} 14 & 51 & 20 \\ 9 & 17 & 3 \\ 3 & 16 & 15 \end{bmatrix}.$$

Example No 4.

An interesting result is obtained multiplying a row matrix by a column matrix:

$$\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 1 \\ -2 \end{bmatrix},$$

$$\mathbf{c} = \mathbf{a}^{\mathsf{T}} \cdot \mathbf{b},$$

$$\mathbf{c} = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \\ -2 \end{bmatrix} = [1 \cdot 3 + 2 \cdot 2 + 3 \cdot 1 + 4 \cdot (-2)] = 2,$$

The matrix **c** with dimensions 1×1 (so it is a scalar) is the result of this operation.

Thus, the vector multiplication $\mathbf{a}^{\mathsf{T}} \cdot \mathbf{b}$ is called scalar multiplication.

The matrix multiplication is not in general the commutative operation which means

$$\mathbf{A}\,\mathbf{B} \neq \mathbf{B}\,\mathbf{A},$$

even if it can be done (it is possible only for quadratic matrices).

We will also give some more definitions concerning matrix multiplication which are worth memorising:

$$(\mathbf{A}\,\mathbf{B})\,\mathbf{C} = \mathbf{A}\,(\mathbf{B}\,\mathbf{C}),$$

$$\mathbf{A}\,(\mathbf{B} + \mathbf{C}) = \mathbf{A}\,\mathbf{B} + \mathbf{A}\,\mathbf{C},$$

$$\mathbf{A}\,\mathbf{I} = \mathbf{I}\,\mathbf{A} = \mathbf{A},$$

$$(\mathbf{A}\,\mathbf{B})^{\mathsf{T}} = \mathbf{B}^{\mathsf{T}}\,\mathbf{A}^{\mathsf{T}}.$$

Determinant of a matrix

A determinant is the scalar function of a square matrix which we write as follows:

$$\det \mathbf{A} = \left| A_{ij} \right|.$$

Calculation of the value of a determinant depends on the summation of products obtained from all permutations of components of the matrix **A**:

$$\det \mathbf{A} = \sum_{p} (-1)^{I_p} A_{1\alpha_1}, A_{2\alpha_2}, A_{3\alpha_3}, \ldots A_{n\alpha_n},$$

where p denotes all permutations, I_p - number of inversions in the permutations.

The value of a determinant can also be calculated by using Laplace's expansion with regard to terms of any rows or columns:

$$\det \mathbf{A} = \sum_{k=1}^{n} A_{mk}\, \overline{A}_{mk} \text{ - development of the row } m \ (1 \le m \le n)$$

or

$$\det \mathbf{A} = \sum_{k=1}^{n} A_{km}\, \overline{A}_{km} \text{ - development of the column } m \ (1 \le m \le n).$$

$\overline{A}_{ij}$ here signifies the algebraic complement of the element A_{ij} of the matrix:

$$\overline{A}_{ij} = (-1)^{i+j} \left| A_{ij}^{*} \right|,$$

where $\left| A_{ij}^{*} \right|$ is the minor of the matrix $\mathbf{A}^{*} = \left[A_{ij}^{*} \right]$ that is to say the determinant of a matrix which is obtained by removing the row i and the column j from the matrix **A**.

Laplace's development should be processed as long as we obtain matrices 2x2 whose determinants can be calculated directly:

$$\det \mathbf{A} = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = A_{11}A_{22} - A_{12}A_{21}.$$

The way of calculating determinants of the matrix 3x3 (Sarrus's rule) is also known as

$$\det \mathbf{B} = \begin{vmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{vmatrix} =$$

$$= B_{11}B_{22}B_{33} + B_{21}B_{32}B_{13} + B_{31}B_{12}B_{23} - B_{31}B_{22}B_{13} - B_{21}B_{12}B_{33} - B_{11}B_{32}B_{23}.$$

Yet it should not be applied to matrices with a greater number of rows and columns.

It is worth memorising the useful relation:

$$\det(\mathbf{A}\,\mathbf{B}) = \det \mathbf{A} \det \mathbf{B},$$

which helps us to determine determinants of products of matrices effectively.

If the determinant of a matrix is equal to zero, then such a matrix is called a singular matrix.

Inverse of a matrix

A matrix satisfying the condition:

$$\mathbf{A}\,\mathbf{A}^{-1} = \mathbf{I}$$

is called the inverse of the matrix **A**.

Components of an inverse matrix can be determined by scaling a transpose matrix of algebraic complements:

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}\overline{\mathbf{A}}^{\mathrm{T}} = \frac{1}{|A_{ij}|}\left[\overline{A}_{ij}\right]^{\mathrm{T}} = \frac{\left[(-1)^{i+j}\left|A_{ij}^{*}\right|\right]^{\mathrm{T}}}{|A_{ij}|},$$

where $\overline{\mathbf{A}} = \left[\overline{A}_{ij}\right]$ is the matrix of algebraic complements: $\overline{\mathbf{A}} = \left[(-1)^{i+j}\left|A_{ij}^{*}\right|\right]$,

$\left|A_{ij}^{*}\right|$ is the minor, that is, the determinant of a matrix which is formed by removing the row i and the column j from the matrix **A**.

It is easy to note that it is impossible to find a matrix which would be the 'inverse' of a singular matrix because it requires dividing by zero.

The matrix $\overline{\mathbf{A}}^{\mathrm{T}}$ is called the joined matrix of the matrix **A**. The joined matrix can be formed for any matrix (even singular).

Example No 5.

We look for the 'inverse' of the matrix:

$$\mathbf{A} = \begin{bmatrix} 9 & 6 & 2 \\ 1 & 9 & 3 \\ 7 & 5 & 3 \end{bmatrix}.$$

First, we calculate the determinant in order to check if the inverse operation is possible. We calculate the determinant of the matrix **A** making use of Sarrus's rule:

$$\det \mathbf{A} = (9\cdot 9\cdot 3) + (1\cdot 5\cdot 2) + (7\cdot 6\cdot 3) - (7\cdot 9\cdot 2) - (1\cdot 6\cdot 3) - (9\cdot 5\cdot 3) = 100\,.$$

We calculate sequencing the algebraic complements:

$$\overline{A}_{11} = (-1)^{1+1}\begin{vmatrix} 9 & 3 \\ 5 & 3 \end{vmatrix} = 12, \qquad \overline{A}_{12} = (-1)^{1+2}\begin{vmatrix} 1 & 3 \\ 7 & 3 \end{vmatrix} = 18,$$

$$\overline{A}_{13} = (-1)^{1+3}\begin{vmatrix} 1 & 9 \\ 7 & 5 \end{vmatrix} = -58, \qquad \overline{A}_{21} = (-1)^{2+1}\begin{vmatrix} 6 & 2 \\ 5 & 3 \end{vmatrix} = -8,$$

$$\overline{A}_{22} = (-1)^{2+2}\begin{vmatrix} 9 & 2 \\ 7 & 3 \end{vmatrix} = 13, \qquad \overline{A}_{23} = (-1)^{2+3}\begin{vmatrix} 9 & 6 \\ 7 & 5 \end{vmatrix} = -3,$$

$$\overline{A}_{31} = (-1)^{3+1}\begin{vmatrix} 6 & 2 \\ 9 & 3 \end{vmatrix} = 0, \qquad \overline{A}_{32} = (-1)^{3+2}\begin{vmatrix} 9 & 2 \\ 1 & 3 \end{vmatrix} = -25,$$

$$\overline{A}_{33} = (-1)^{3+3}\begin{vmatrix} 9 & 6 \\ 1 & 9 \end{vmatrix} = 75,$$

from which we have

$$\mathbf{A}^{-1} = \begin{bmatrix} 0.12 & -0.08 & 0.0 \\ 0.18 & 0.13 & -0.25 \\ -0.58 & -0.03 & 0.75 \end{bmatrix}.$$

Decomposition of a matrix into triangular matrices

The nonsigular matrix **A** can be broken down into the product of triangular matrices:

$$\mathbf{A} = \mathbf{L}\,\mathbf{U},$$

where **L** is the lower triangular matrix and **U** is the upper triangular matrix. Such a process is called either matrix triangulation or decomposition or factorisation.

The decomposition method was originated by M.H.Doolittle (1878) and later it was reconfirmed by findings of several scientists like Cholesky (1916), A.C.Aitken (1932), T.Banachewicz (1938) and P.D.Crout (1941). The Cholesky method was described by Benoit in 1924.

The components of the triangular matrix **L** and **U** can be calculated using the procedures proposed by Crout or Banachewicz:

$$L_{ii} = 1, \; i = 1 \ldots n,$$

$$U_{ij} = A_{ij} - \sum_{k=1}^{i-1} L_{ik} U_{kj}, \; j = i \ldots n,$$

$$L_{ij} = \frac{1}{U_{jj}}\left(A_{ij} - \sum_{k=1}^{j-1} L_{ik} U_{kj} \right), \; i = j \ldots n.$$

Calculation of components is done alternatively for rows of the matrix **U** and columns of the matrix **L** (the Crout method) or in succession the row of the matrix **U** and then the row of the matrix **L** (the Banachewicz method [18]).

Decomposition into triangular matrices is very important in practice because it is applied as the effective method of solving sets of linear equations.

The solution of the set of equations

$$\mathbf{A}\,\mathbf{x} = \mathbf{y}$$

can be obtained in two stages. At the first stage we apply substitutions $\mathbf{A} = \mathbf{L}\,\mathbf{U}$ and $\mathbf{U}\,\mathbf{x} = \mathbf{z}$ which simplify the set of equations to the form:

$$\mathbf{L}(\mathbf{U}\,\mathbf{x}) = \mathbf{y} \rightarrow \mathbf{L}\,\mathbf{z} = \mathbf{y}$$

which simplifies solving

$$z_1 = \frac{y_1}{L_{11}},$$

$$z_2 = (y_2 - L_{21}z_1)\frac{1}{L_{22}}, \text{ etc.,}$$

$$z_i = \left(y_i - \sum_{k=1}^{i-1} L_{ik} z_k\right)\frac{1}{L_{ii}}.$$

The applied procedure is called here forward elimination because we calculate consecutively the unknowns z_1, z_2 ... z_i ... z_n.

The second stage depends on the determination of unknown values from equations

$$\mathbf{U}\,\mathbf{x} = \mathbf{z},$$

which is done similarly to the previously used method but we have applied back substitution starting from the last component:

$$x_n = \frac{z_{nn}}{U_{nn}},$$

$$x_{n-1} = (z_{n-1} - U_{n-1n}x_n)\frac{1}{U_{n-1n-1}}, \text{ etc.,}$$

$$x_i = \left(z_i - \sum_{k=i+1}^{n} U_{ik} x_k\right)\frac{1}{L_{ii}}.$$

Time to solve a set of equations by this method is proportional to $n^3/3$, where n is the number of equations. The number $T_D = n^3/3$ is called the cost of Doolittle's method and is the estimated number of multiplication and division operations which should be done in order to solve a set of equations.

Triangularization of symmetric matrices

If the square matrix is symmetric (obviously not singular) decomposition given in the previous section can be simplified even more noting that:

$$\mathbf{A} = \mathbf{L}\,\mathbf{L}^{\mathrm{T}} \text{ or } \mathbf{A} = \mathbf{U}^{\mathrm{T}}\mathbf{U}.$$

The algorithm of the decomposition of the symmetric matrix **A** into triangular matrices was published for the first time by Cholesky (in 1916) and then independently by Banachewicz (in 1938). This method is usually called the Cholesky method. In Poland the name the Banachewicz-Cholesky method is used in scientific publications.

Components of a triangular lower matrix obtained by this method are equal to:

$$L_{ij} = 0 \text{ for } j > i,$$

$$L_{ii} = \sqrt{A_{ii} - \sum_{k=1}^{i-1} L_{ik}^2}\ ,$$

$$L_{ij} = \left(A_{ij} - \sum_{k=1}^{j-1} L_{ik} L_{jk} \right) \frac{1}{L_{jj}} \text{ for } j < i.$$

In the above equations defining the components lying on the main diagonal of the matrix **L** a square root is applied. The term under the root can certainly be negative and then components of the matrix **L** are complex. It can be proved [7] that for positively defined symmetric matrices the components L_{ii} are always real numbers.

Time of the decomposition of a symmetric matrix obtained by the Banachewicz-Cholesky method is proportional to $T_{B\text{-}CH} = n^3/6$.

Example No 6.

Using the Banachewicz-Cholesky method, find the triangular lower matrix **L** for which $\mathbf{A} = \mathbf{L}\mathbf{L}^{\mathrm{T}}$

$$\mathbf{A} = \begin{bmatrix} 10 & 1 & 2 & -1 \\ 1 & 15 & 2 & -3 \\ 2 & 2 & 13 & 4 \\ -1 & -3 & 4 & 12 \end{bmatrix}.$$

We determine particular components of the triangular lower matrix **L** which are different from zero:

$$L_{11} = \sqrt{A_{11}} = \sqrt{10} = 3.16228,$$

$$L_{21} = A_{21}\frac{1}{L_{11}} = \frac{1}{\sqrt{10}} = 0.32623,$$

$$L_{22} = \sqrt{A_{22} - L_{12}^2} = \sqrt{15 - \left(\frac{1}{\sqrt{10}}\right)^2} = 3.86005,$$

$$L_{31} = A_{31}\frac{1}{L_{11}} = \frac{2}{\sqrt{10}} = 0.63246\,,$$

$$L_{32} = \left(A_{32} - L_{31}L_{21}\right)\frac{1}{L_{22}} = \left(2 - \frac{2}{\sqrt{10}}\frac{1}{\sqrt{10}}\right)\frac{1}{\sqrt{14.9}} = 0.46631\,,$$

$$L_{33} = \sqrt{A_{33} - \left(L_{31}^2 + L_{32}^2\right)} = \sqrt{13 - \left(\left(\frac{2}{\sqrt{10}}\right)^2 + \left(\frac{1.8}{\sqrt{14.9}}\right)^2\right)} = 3.51888\,,$$

$$L_{41} = A_{41}\frac{1}{L_{11}} = -0.31623\,,$$

$$L_{42} = \left(A_{42} - L_{41}L_{21}\right)\frac{1}{L_{22}} = -0.75129\,,$$

$$L_{43} = \left(A_{43} - \left(L_{41}L_{31} + L_{42}L_{32}\right)\right)\frac{1}{L_{33}} = 1.29312\,,$$

$$L_{44} = \sqrt{A_{44} - \left(L_{41}^2 + L_{42}^2 + L_{43}^2\right)} = 3.10860\,.$$

$$\mathbf{L} = \begin{bmatrix} 3.16228 & 0 & 0 & 0 \\ 0.31623 & 3.86005 & 0 & 0 \\ 0.63246 & 0.46631 & 3.51888 & 0 \\ -0.31623 & -0.75129 & 1.29312 & 3.10860 \end{bmatrix}$$

Orthogonal matrices

There is a group of matrices having the property:

$$\mathbf{A}^{-1} = \mathbf{A}^{\mathrm{T}}$$

which enormously simplifies solving a set of equations. We say that such matrices are orthogonal matrices. This property is shown by the transformation matrices for vectors:

$$\mathbf{R} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix},$$

where $c = \cos\alpha$, $s = \sin\alpha$, and α is a rotation angle.

We check the orthogonality of this matrix by the equation $\mathbf{R}\,\mathbf{R}^{\mathrm{T}} = \mathbf{I}$:

$$\begin{bmatrix} c & -s \\ s & c \end{bmatrix} \times \begin{bmatrix} c & s \\ -s & c \end{bmatrix} = \begin{bmatrix} c^2 + s^2 & cs - sc \\ sc - cs & c^2 + s^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We use this property of the transformation matrix in some chapters of this book.

Appendix 2. Methods of solving large sets of linear equations

Sets of equations occurring in the finite element method are characterised by large, rare and positive-definite matrices. Methods of solving sets of equations of such a type of matrices differ slightly from the ways of solving any other sets and all mentioned above methods have to consider ways of storing of matrices in the computer memory.

Methods of storage of stiffness matrices

Not a very complex exercise on the use of the finite element method, for example a shell structure, generates a set of equations of the order of unknown parameters 1000÷10000. The square matrix of this set of equations becomes a banded symmetric matrix with suitable numbering degrees of freedom (there are very complex procedures of numbering of degrees of freedom using the graph theory). Hence only half of this band is enough to be memorised in order to make the reconstruction of the whole information written in the stiffness matrix of a structure possible.

The simplest method of saving computer memory is recording the upper or lower matrix half bands in the rectangular table shown in Fig.A2.1.

It changes the location of matrix elements in the table so that elements from the main diagonal are located in the first column of the band and, for example, the element which originally was in the row i and the column j is still in the same row but in the column k. The new value of a column index should be calculated on the basis of a simple relation:

$$k = j - i + 1$$

before getting a necessary component. Thus, we have $B_{ik}=A_{ij}$ for $j \geq i$. The half band width p for typical matrices is usually smaller by one order of value than the dimension n. Hence the lower triangle of the table **B** which is always 'empty', does not have any particular significance for saving the core memory.

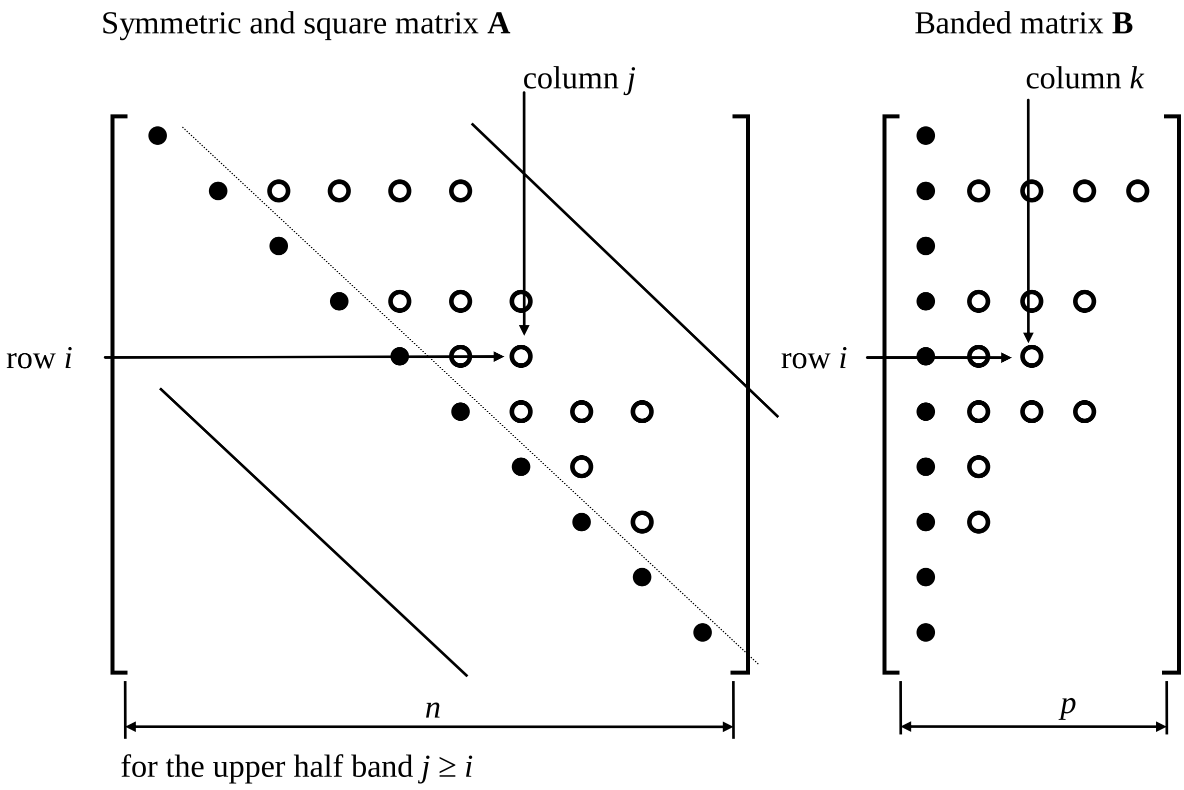


Fig.A2.1

Another economical method is *the sky-line method* which depends on memorising only these parts of rows (or columns) of the upper or lower half band which lie between the main diagonal and the last non-zero elements of the table (Fig.A2.2).

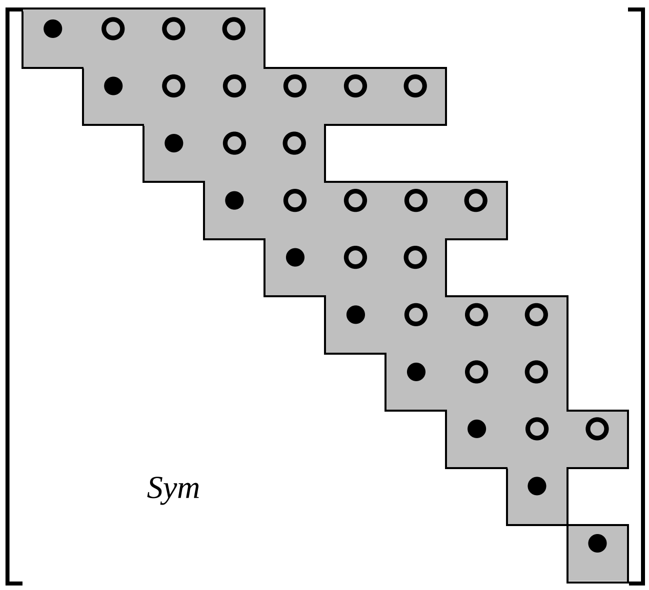
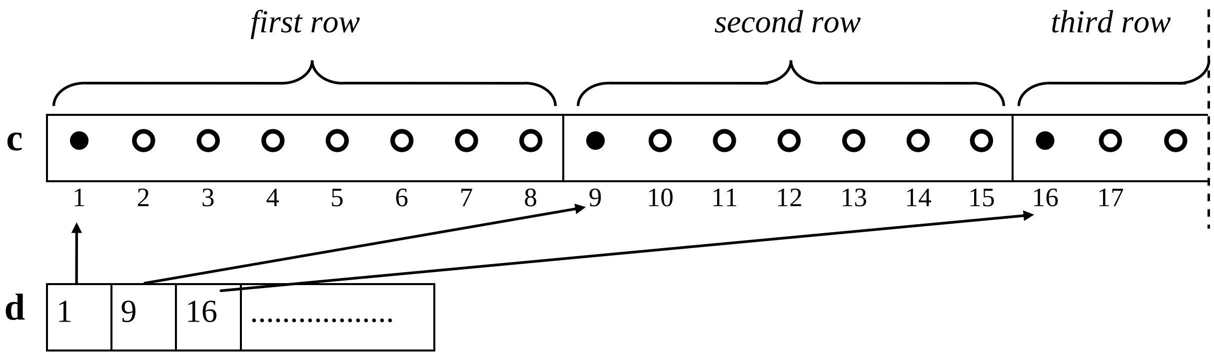


Fig.A2.2

The memorising area is shaded in Fig.A2.2.

Such a method of storing a matrix is possible thanks to the fact that non-zero elements of a triangular matrix never appear in areas lying behind the final non-zero components in rows when the decomposition of the matrix takes place. It is very important because procedures which memorise the matrix **L** in the same table in which the stiffness matrix has been memorised are usually applied to the FEM algorithm. The irregular shapes of the area shown in Fig.A2.2 prevent arranging data in the form of a two-dimensional table. Thus, two one-dimensional tables (vectors) are applied to the sky-line method. One of them stores real numbers which are components of a matrix and the other one stores indexes of the first terms of the successive rows of the matrix (Fig.A2.3).



$A_{ij} = C_k, \qquad k = \mathbf{d}[i]+j-i$

Fig.A2.3

This method is widely applied though it requires fairly complex operations while building a matrix and solving a set of equations (continual calculation of indeces) because it ensures very effective exploitation of the computer memory.

The Gauss elimination method

The Gauss elimination method (in different variants) is one of the most often applied methods of solving sets of linear equations of the type $\mathbf{A}\,\mathbf{x} = \mathbf{y}$ where the matrix **A** is quadratic and singular.

We start solving it from the transformation of the first equation:

$$x_1 = \frac{1}{A_{11}}\left(y_1 - \sum_{k=2}^{n} A_{1k} x_k\right)$$

and the insertion of so determined unknown into other equations. It causes the elimination of the first column in the equations 2 to n (Fig.A2.4).

$$\left[\begin{array}{c|c} 1 & \mathbf{A}_{1k}^{(1)} \quad \mathbf{A}_{2k}^{(1)} \ldots\ldots \\ \hline \mathbf{0} & \mathbf{A}^{(1)} \\ & \end{array}\right] \left[\begin{array}{c} x_1 \\ \hline \mathbf{x}^{(1)} \\ \end{array}\right] \left[\begin{array}{c} y_1/A_{11} \\ \hline \mathbf{y}^{(1)} \\ \end{array}\right]$$

Fig.A2.4. A set of linear equations after the first elimination.

We repeat this operation for the matrix $\mathbf{A}^{(1)}$ with dimensions (n-1)x(n-1) obtaining the matrix $\mathbf{A}^{(2)}$ with dimensions (n-2)x(n-2), etc. We carry on transformations as long as we obtain an equation with one unknown parameter:

$$A_{nn}^{(n-1)} x_n = y_n^{(n-1)},$$

from which we determine x_n.

We can say that the Gauss elimination depends on such transformation of a matrix of a set of linear equations which leads to building a set of equations with an upper triangular matrix:

$$\mathbf{Ax} = \mathbf{y} \xrightarrow{the Gauss elimination} \mathbf{Ux} = \mathbf{y}^*,$$

which we solve by applying the back substitution method described in Appendix 1. The cost of the Gauss method is equal to $n^3/3$ and can really be proved (comp. [2]) that a cheaper algorithm cannot be found.

While eliminating unknown parameters the division operation by the diagonal component of the matrix **A** continually appears in those transformations. It can happen that $A_{ii}^{(k)}$ will be equal to zero or close to zero even for a nonsingular matrix. It can prevent obtaining the solution or leads to serious numerical errors. Such a situation can be avoided by conducting the elimination process in a different order. The change in the choice order of unknown parameters for the elimination enables to find such a diagonal component which is the biggest one in the matrix $A^{(k)}$ and to minimise the number of numerical errors.

The variant of the Gauss elimination with the choice of a middle element is called the Gauss-Jordan method. It enables to obtain a solution with an insignificant error even for slightly conditioned sets of equations, that is for sets with the determinant of the matrix **A** close to zero.

Part of the source code (in the PASCAL language) solving sets of linear equations (the Gauss procedure) presented in the following section is an example of the realisation of the Gauss algorithm.

The Gauss-Seidel iterative method

The Gauss-Seidel iterative method is based on the assumption that the diagonal components of a matrix are considerably larger than components lying behind the diagonal. Thanks to it we can calculate

$$x_1 = \frac{1}{A_{11}}\left(y_1 - \sum_{k=2}^{n} A_{1k} x_k \right),$$

with the initial assumption that $x_k = 0$ for $k = 2 \dots n$. We repeat this approximation for other unknown values:

$$x_i = \frac{1}{A_{ii}}\left(y_i - S_{iL} - S_{iR} \right),$$

where S_{iL} is the sum of all products of terms lying on the left side of x_i and suitable unknown values and S_{iR} is the sum of products of terms lying the right side of x_i and suitable unknown values:

$$S_{iL} = \sum_{k=1}^{i-1} A_{ik} x_k ,$$

$$S_{iR} = \sum_{k=i+1}^{n} A_{ik} x_k .$$

Successive approximation of unknown values done by this method is concurrent when a set of equations is well conditioned, which means that terms lying on the diagonal are larger than components lying behind it. The stiffness matrices of the finite element method are built in such a way. The Seidel modification of this method depends on the consideration of current unknown values while the iteration m which signifies the sum S_{iL} is calculated using unknown parameters during the iteration m, and the sum S_{iR} is calculated on the basis of unknown values determined in the previous iteration (m-1):

$$x_i^{(m)} = \frac{1}{A_{ii}}\left(y_i - S_{iL}^{(m)} - S_{iR}^{(m-1)} \right),$$

where $S_{iL}^{(m)} = \sum_{k=1}^{i-1} A_{ik} x_k^{(m)}$, $S_{iR}^{(m-1)} = \sum_{k=i+1}^{n} A_{ik} x_k^{(m-1)}$, $x_k^{(m)}$ is the value of the unknown x_k determined in the iteration m.

After every iterative step we calculate the difference $\Delta_i^{(m)} = x_i^{(m)} - x_i^{(m-1)}$ which allows to check the concurrence of the process. Iterations can be broken when $\mathrm{Max}\left(\left|\Delta_\mathrm{i}\right|\right) < \varepsilon$, which means that the biggest difference is smaller than the permissible error of calculation. For large sets of equations we can often obtain the solution of a set of equations by the Gauss-Seidel method faster than by using the closed method (for example the Gauss-Jordan method).

The Aitken overrelaxation method

We note in the Gauss-Seidel iterative process that

$$x_i^{(m)} = x_i^{(m-1)} + \Delta_i^{(m)},$$

where the unknown value approaches the exact value with the step $\Delta_i^{(m)}$. Aitken noted that velocity of the process can be increased (that is, the number of necessary iterations can be decreased) if we calculate

$$x_i^{(m)} = x_i^{(m-1)}\omega\,\Delta_i^{(m)},$$

where ω is a overrelaxation coefficient. The value of this coefficient should be fitted on the basis of numerical experiments and it should be contained within the range $\langle 1.0 \div 2.0 \rangle$. Our calculations show that for the static problem of a 3D truss the optimal value of the overrelaxation coefficient is equal to 1.26.

Other methods of solving large sets of equations

Sets of equations of the finite element method are very often solved by methods depending on matrix decomposition, for example, the Banachewicz-Cholesky method presented in Appendix 1. The cost of this method is proportional to $n^3/6$ for the full symmetric matrix and it is equal to $np^2/6$, where p is the half band width of the matrix for banded matrices used in FEM problems.

Apart from the Banachewicz-Cholesky method some other methods of decomposition are also applied, for example, the Crout method consisting in splitting the matrix $\mathbf{A}$ into three matrices:

$$\mathbf{A} = \mathbf{L}\,\mathbf{D}\,\mathbf{L}^\mathrm{T},$$

where $\mathbf{D}$ is the diagonal matrix which means that it contains non-zero components only on the main diagonal. Such a type of distribution is not as unique as the Banachewich-Cholesky distribution, thus, the diagonal components of the matrix $\mathbf{L}$ are chosen so that they are equal to 1. The Crout decomposition is often applied to solving FEM problems, and particularly in nonlinear problems where the stiffness matrix is not always positive-definite. In this case the Banachewicz-Cholesky method leads to the formation of the matrix **L** with complex numbers. It results from the fact that diagonal terms are calculated there by extracting roots. In the Crout method we always obtain a matrix with real components [1], [2] .

The Crout decomposition leads to the following relation:

$$D_{ii} = A_{ii} - \sum_{k=1}^{i-1} L_{ik}^2 D_{kk}$$

$$D_{ij} = 0 \ \text{for}\ j \neq i,$$

$$L_{ij} = 0 \ \text{for}\ j > i,$$

$$L_{ii} = 1.0\,,$$

$$L_{ij} = \frac{1}{D_{jj}}\left(A_{ij} - \sum_{k=1}^{j-1} L_{ik} L_{jk} D_{kik} \right) \ \text{for}\ j < i,$$

The cost of matrix decomposition by the Crout method is proportional to $n^3/6$ for full matrices similarly to the cost of the process by the Banachewicz-Cholesky method.

Appendix 3. Stiffness of torsion frame elements

The problem of torsion bars is very important in practice. The determination of the bar stiffness in the process of torsion is necessary to determine components of stiffness indices of 3D frame elements (comp. Chapter 5). The problem of determination of stress and stiffness of a bar with a circular symmetric cross section (Fig.A3.1) was solved by Coulomb at the end of 18th century [20].

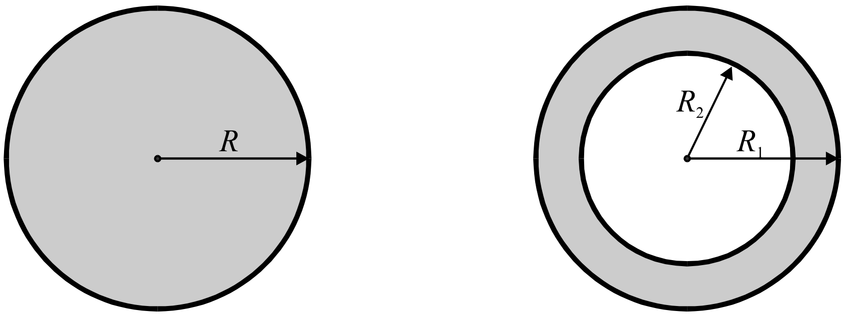


Fig.A3.1

A. Circular cross section

In case of circular cross sections their torsion stiffness is equal to the polar moment of inertia and:

$C = J_o = \dfrac{\pi R^4}{2}$ for a full circular cross section

and

$C = J_o = \dfrac{\pi}{2}\left(R_1^4 - R_2^4\right)$ for a pipe cross section.

Thus, the dependence between the torsion moment M_s and a unit angle of a cross section rotation is equal to

$M_s = CG\vartheta$.

The problem of determination of stiffness and stress in a torsion bar with any cross section was solved by de Saint-Venant in the middle of 19th century. He assumed that non-circular cross sections undergo deplanation. The determination of a warping function requires solving a harmonic differential equation:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0.$$

Many ways of solving this problem for different cross sections can be found in the book written by P.S.Timoshenko and J.N.Goodier [20] and another one written by

M.T.Huber [8]. In this Appendix we give ready made solutions for a few different from the technical point of view cross sections.

B. An elliptic cross section

This problem was solved by de Saint-Venant in 1855.

$$C = \pi \frac{a^3 b^3}{a^2 + b^2},$$

where *a* and *b* are half axes of an ellipse.

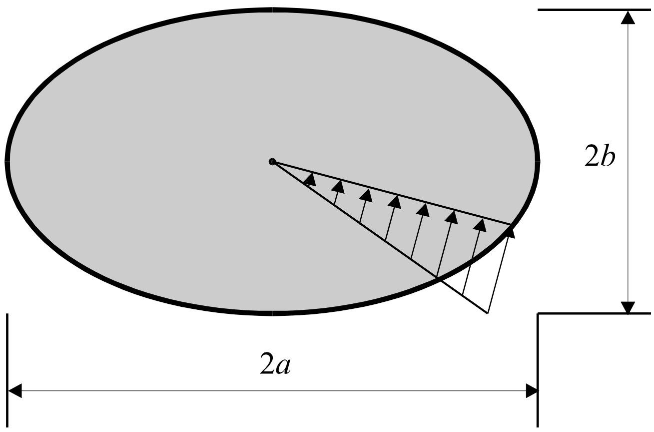


Fig.A3.2

C. An equilateral triangle

This problem was solved by de Saint-Venant in 1855.

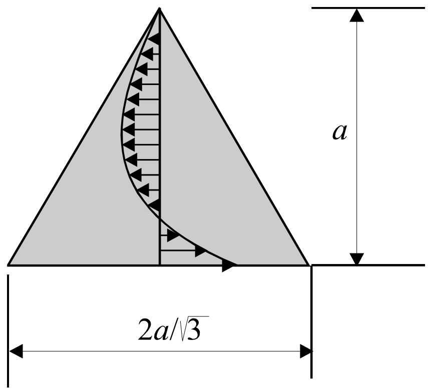


Fig.A3.3

$$C = \frac{a^4 \sqrt{3}}{45}$$

D. A rectangular cross section

That problem was solved by de Saint-Venant in 1856.

$$\frac{a}{b} \le 1$$

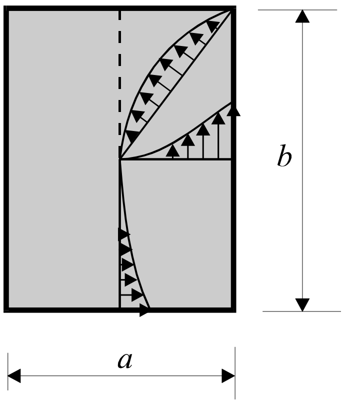


Fig.A3.4

$$C = k\left(\frac{a}{b}\right)a^3 b\,,$$

where $k\left(\frac{a}{b}\right) = \frac{1}{3} - \frac{64}{\pi^5}\frac{a}{b}\sum_{n=1,3,5}^{\infty}\frac{1}{n^5}\,\mathrm{tgh}\frac{n\pi b}{2a}$

Proper approximation can be obtained by using the formula of C.Weber [13,15]:

a) $$k\left(\frac{a}{b}\right) \approx \frac{1}{3}\left[1 - 0.63\frac{a}{b} + 0.052\left(\frac{a}{b}\right)^5\right],$$

or its modification

b) $$k\left(\frac{a}{b}\right) \approx \frac{1}{3} - 0.21\frac{a}{b}\left(1 - \frac{a^4}{12b^4}\right),$$

giving the value which differs from the exact value not more than by 0.55% (at $\frac{a}{b} \approx 0.875$).

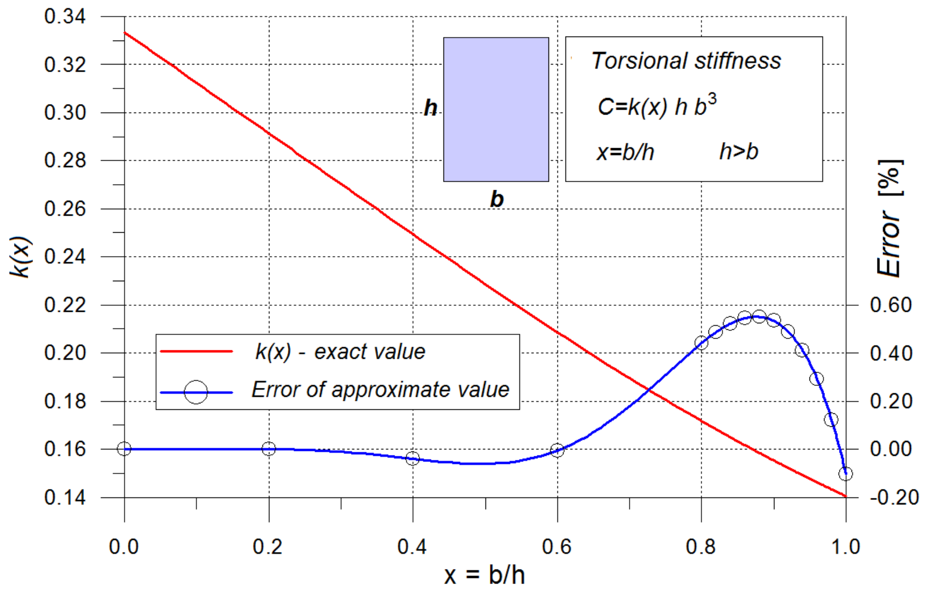


Fig.A3.5

The graph shows the dependence $k\left(\dfrac{a}{b}\right)$ which can be used for approximate determination of stiffness of a rectangular cross section (Fig.A3.5).

E. A circular segment

This problem was solved by de Saint-Venant in 1878.

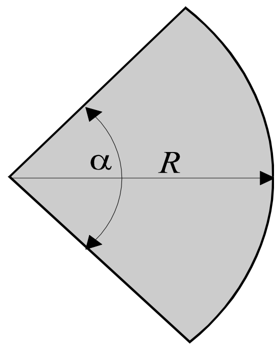


Fig.A3.6

$C = k(\alpha)R^4$,

where $k(\alpha)$ is the coefficient calculated on the basis of the equation:

$$k(\alpha) = \int_{-\alpha/2}^{\alpha/2}\int_{0}^{R}\left[-r^2\left(1-\frac{\cos 2\varphi}{\cos\alpha}\right)+\frac{16r^2\alpha^2}{\pi^3}\sum_{n=1.3.5}^{\infty}(-1)^{\frac{n+1}{2}}\left(\frac{r}{R}\right)^{\frac{n\pi}{\alpha}}\frac{\cos\dfrac{n\pi\varphi}{\alpha}}{n\left(n\dfrac{2\alpha}{\pi}\right)\left(n-\dfrac{2\alpha}{\pi}\right)}\right]rd\varphi dr\,.$$

We give the values of this coefficient for a few values of the angle α in the table below:

α	π/4	π/3	π/2	2π/3	π	3π/2	5π/3	2π
k	0.0181	0.0349	0.0825	0.148	0.296	0.572	0.672	0.878

F. An isosceles right-angled triangle

The above problem was solved by Galerkin in 1919.

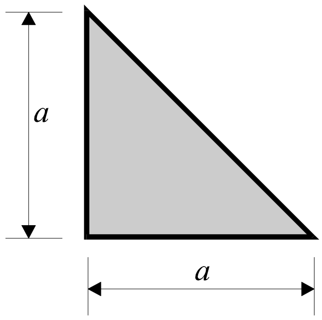


Fig.A3.7

$$C = \frac{a^4}{38.3}$$

G. A regular hexagon

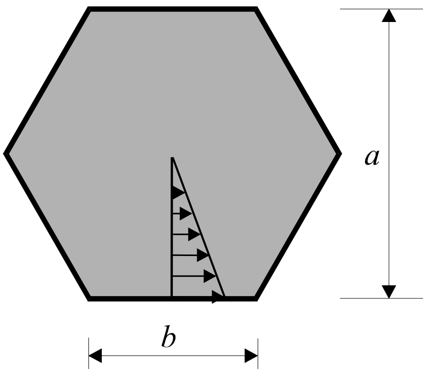


Fig.A3.8

$$J_o = \frac{5\sqrt{3}}{8} b^4 \ ; \ A = \frac{3\sqrt{3}}{2} b^2 \ ;$$

$$C = 1.0366\, b^4 \ ;$$

$$\beta = \frac{A^4}{CJ_o} = 40.603 \ ; \ \tau_{\max} = 0.564 b^2 .$$

H. A thin-walled pipe with any cross section

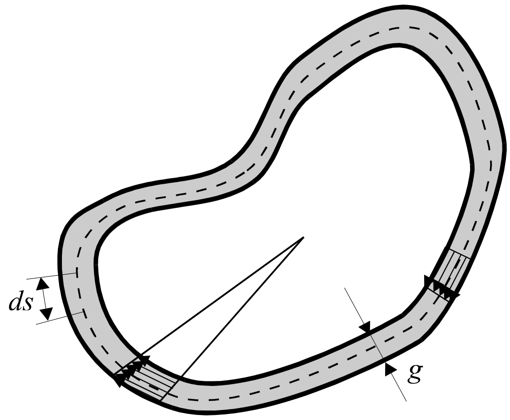


Fig.A3.9

$$C = \frac{4A_o^2}{\oint\limits_S \frac{ds}{g(\alpha)}},$$

where A_o is the surface of a figure limited by a line dividing the thickness of a pipe wall into halves. Integration should be done along the circuit S of this figure.

I. A thin-walled pipe cut along generating line

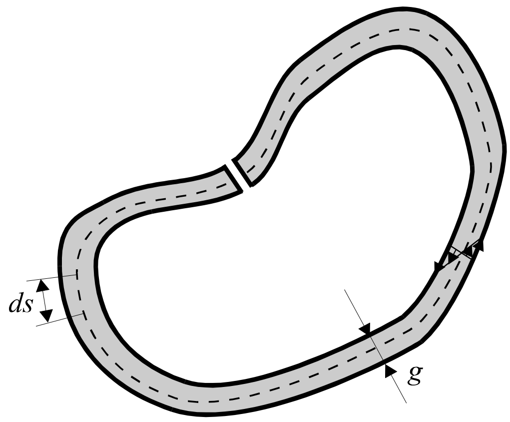


Fig.A3.10

$$C = \frac{1}{3}\int\limits_S g^3 ds\,.$$

It is interesting to notice that stiffness does not depend on the shape of a cross section but it depends on its thickness and circuit S.

J. Cross sections composed of thin-walled rectangles

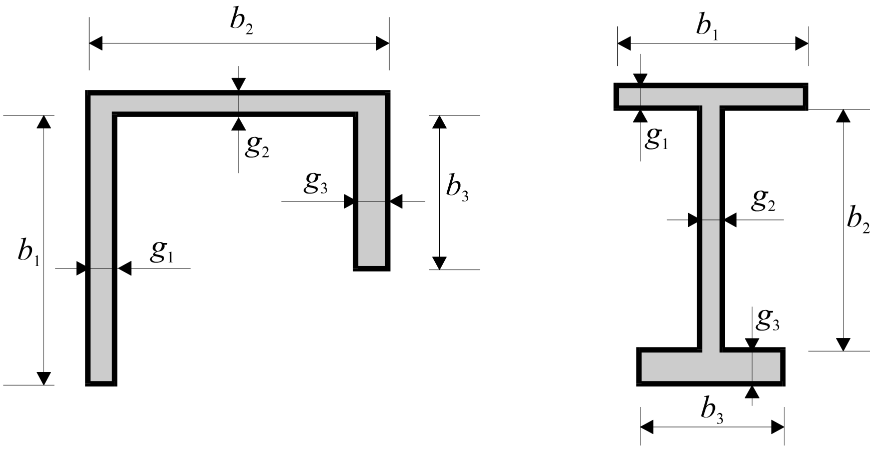


Fig.A3.11

$$C = \frac{1}{3}\sum_{i=1}^{n} g_i^3 b_i$$

Comparing coefficient $\frac{1}{3}$ in the above formula with the graph shown in Fig.A3.5, we note that stiffness is always overevaluated. For a cross section composed of rectangles with the same thickness more exact results are obtained by using the formula for rectangles (example D) where we substitute *g* for *a* and the length of a circuit of the middle line of a cross section is substituted for

$$b = \sum_{i=1}^{n} b_i \ .$$

K. A thick-walled pipe cut along generating line

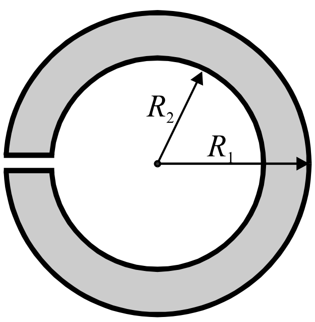


Fig.A3.12

$$C = \frac{\pi}{2}\left[R_2^4 - R_1^4 - \frac{\left(R_2^2 - R_1^2\right)^2}{\ln\left(\frac{R_1}{R_0}\right)} \right]$$

L. Other cross sections with crowned contour

On the basis of many exact solutions de Saint-Venant proposed to determine the torsion stiffness from the approximate formula:

$$C = \frac{A^4}{4\pi^2 J_o},$$

where A is the surface of a cross section and J_o is the center moment of inertia.

The above formula is exact for an ellipse. Generalising it, we write

$$C = \frac{A^4}{\beta J_o},$$

where β is the coefficient depending on the shape of a cross section. The table below in which you can find several different values of the coefficient β can be helpful as a reference.

Section	Circle, ellipse	Equilateral triangle	Rectangle				Circular segment		Isosceles right-angled triangle	Regular hexagon
			1:1	2:3	1:2	1:4	α=π/2	α=π		
Example	A, B	C	D	D	D	D	E	E	F	G
β	$4\pi^2$= 39.478	45	42.674	42.438	41.976	40.221	42.022	40.935	43.088	40.603

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